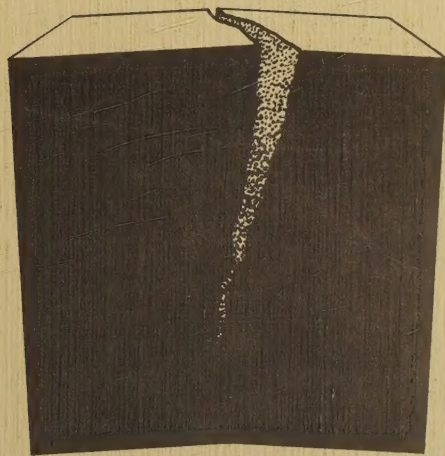


Equations of Balance and Related Problems in Continuum Fracture Mechanics



Per Lidström





EQUATIONS OF BALANCE AND RELATED PROBLEMS

IN

CONTINUUM FRACTURE MECHANICS

BY

PER LIDSTRÖM

Papers summarized in this thesis:

- I P. Lidström, Equations of Balance in Continuum Fracture Mechanics.
- II P. Lidström, Crack Propagation in a Hyperelastic Material and the Rice Fracture Criterion.
- III P. Lidström, The Quasi-static Energy Release Rate and the J-integral in the Linearized Theory of Elasticity.
- IV P. Lidström, On the Uniqueness for Plane Crack Problems in Linear Elastostatics.

Technical reports. Department of Mechanics,
University of Lund, Lund 1985.

INTRODUCTION AND SUMMARY OF PAPERS

Consider a body of homogeneous solid material subject to a system of forces applied at its boundary. This arrangement will induce a state of stress in the body. The character of the stress-field will depend on the material of the body, the properties of the force-system and the form of the boundary-surface of the body. Abrupt variations in the geometry of the boundary, like sharp edges and corners, may have a great influence on the neighbouring stresses. For instance, we expect the stress-level at corner c (see Figure 1 below) to be comparatively high. Moreover, we expect the body to start cracking precisely from this corner if the applied forces are sufficiently large.

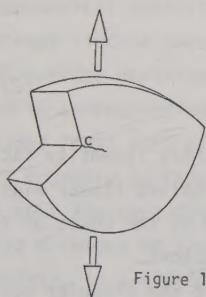


Figure 1

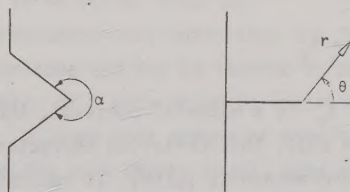


Figure 2

Mathematically speaking this so-called notch-stress at c corresponds, under specific assumptions, to a singularity of the solutions to elliptic boundary value problems at re-entrant (concave) corners of the given domain (Kondratev [1]).

For the bi-harmonic operator in a two-dimensional domain, containing a re-entrant corner with internal angle $\alpha = 2\pi$ (see Figure 2 above), it has been shown by M.L. Williams ([2]) that, in terms of local polar coordinates (r, θ) centered at the corner, the solution has an $O(r^{\frac{3}{2}})$ singularity, so that its second order derivatives are asymptotically proportional to $r^{-\frac{1}{2}}$ (as $r \rightarrow 0$). This gives rise to the well-known square-root singularity of stress at the tip of a crack in linearized elasto-statics.

The distribution of stress and displacement close to the crack-tip in an elastic body in static equilibrium has been analysed, on the basis of the linearized theory, for a great variety of geometries and loading conditions (Sneddon & Lowengrub [3], Sih & Liebowitz [4]). In most of these cases the singular stresses at the crack-tip have the form

$$\sigma_{ij} = \sigma_{ij}(r, \theta) \sim \frac{K}{\sqrt{2\pi r}} \hat{\sigma}_{ij}(\theta), \quad r \rightarrow 0 \quad (1)$$

where $\hat{\sigma}_{ij}$ represents the angular variation of the stress-field and K , the stress intensity factor, is a constant depending on the geometry and the applied load.

It has been argued, from consideration of inter-molecular forces, that unbounded stress at the crack-tip is "un-physical" since no real material can sustain infinite stress. The equilibrium solution assumed would then be impossible or at least useless for practical purposes. G.R. Irwin, however, suggested that the stress intensity factor (or "the force tending to cause crack extension" in his vocabulary) may be used in a criterion for the onset of brittle fracture. According to this criterion the crack starts to propagate when

$$K = K_c \quad (2)$$

where K_c is a material constant, the critical stress intensity factor, (Irwin [5]). This criterion characterizes the so-called linear elastic fracture mechanics (LEFM). It reduces the problem of the onset of fracture to the problem of calculating stress intensity factors.

Another and perhaps even more serious objection to the singular solution is raised by the fact that the unbounded displacement gradients in the elasto-static field are inconsistent with the linearization process. The conjecture that the singularities are introduced in the linearization procedure and that any large deformation theory will give finite stresses at the crack-tip ([4]) is refuted by the work of J.K. Knowles and E. Sternberg. In a number of investigations, carried out within the fully non-linear theory of homogeneous, isotropic, incompressible elastic solids, they have demonstrated the existence of materials that, besides giving rise to unbounded stresses at the crack-tip (some stress-components may be bounded), also produce another type of singularity in the region close to the crack-tip, the so-called elasto-static shock ([6,7]).

Several models have been constructed with the aim to eliminate the singularities, for instance the cohesive zone theory (Prandtl [8], Barenblatt [9], Strifors [10,11], Gurtin [12]) and a non-local theory (Eringen, Speziale & Kim [13]). These efforts to try to get rid of the singularities usually introduce new conceptual and mathematical difficulties.

The elasto-static field, referred to above, may be considerably changed if

we introduce some elastic-plastic model of the material.

Judging by the investigations undertaken so far it seems likely that the very special form of the boundary surface arising in crack-problems will strongly effect stresses and strains in the body and that we may have to reckon with different kinds of singular behaviour.

The boundary surface will influence the state of affairs in yet another way. From results in solid state physics and surface science, based on molecular considerations, it is known that the boundary surface of a body exhibits properties that differ significantly from those associated with the bulk of the body. That the macroscopic effects caused by these differences may be successfully modelled within a continuum theory has been emphasized by C. Herring ([14]). Thus the introduction of macroscopic thermo-mechanical quantities like surface stress, surface energy and surface entropy finds a physical support in molecular theories.

Although surface distributions of thermo-mechanical quantities are "small" compared to corresponding bulk distributions and may be ignored in many discussions (that part of the free energy of a body which is distributed over its boundary surface is typical of some nine orders of magnitude smaller than the part distributed over the bulk of the body), they are needed in a theory of fracture. In fact, without them, it is not possible to formulate, in a closed and consistent way, continuum mechanical balance laws for a fracturing body. A.A. Griffith, in his classical work: The phenomena of rupture and flow in solids (1920), was the first to realize this. The main problem in Griffith [15] concerns the stability of a pre-existing crack in a body of brittle material at given loading conditions and uniform temperature, that is, the question of whether the crack tends to propagate or not, given the external loading on the body. In his treatment of the problem Griffith considered the free energy of the system consisting of the body and the loading and its dependence on a parameter ℓ representing the crack-length. The free energy of the system W is, according to Griffith, equal to the sum of the potential energy of the external loading, the elastic free energy of the body (in the equilibrium configuration corresponding to the given loading and temperature) and the contribution to the free energy from the boundary surface of the body, including surfaces created by fracture. Griffith then proposed that fracture will only occur if

$$\frac{\partial W}{\partial \ell} < 0$$

(3)

In an equivalent formulation this may be stated: A necessary condition for fracture is that, in a "small" increase of crack-length, the work done by the loading forces on the body must exceed the sum of the increase in elastic free energy and surface free energy. It turns out that this gives a fracture criterion equivalent to the one stated in (2).

Griffith identified the surface free energy (per unit area in the present configuration) with surface tension, a material function depending only on temperature. This identification may be valid for particular materials but is incorrect in general (cf. the discussion in [14]).

A vague conception of surface energy has been prevailing in fracture mechanics circles ever since Griffith. In the literature it is referred to as "the surface tension", "the specific fracture energy", "the apparent fracture energy" etc. and it is frequently interpreted as the energy required for breaking atomic bonds in the formation of a unit area of new surface. However, in some investigations, it may also include a part corresponding to dissipative mechanisms close to the crack-tip. See Sih [16] and Eftis & Liebowitz [17] for a critical examination of these matters.

If the boundary surface of the body is endowed with energy it ought to possess a state of stress as well, formally similar to the kind known from shell theory. This fact is usually ignored in fracture mechanics where newly created crack-surfaces are assumed to be stress-free.

It is reasonable to believe that the effect of the surface distribution of stress on the overall deformation of the body will, in general, be small (cf. Nicolson [18]). However, it is not obvious how it will influence the conditions at the crack-tip where we may have to allow for singular behaviour of the fields participating in the balance of momentum and moment of momentum.

From everyday experience we conclude that crack propagation is an irreversible process. We thus expect fracture to be accompanied with irreversible entropy production. This production is distributed over the volume of the body and over its boundary surface, but the primary source, at crack-growth, should be located at the crack-front.

Though entropy considerations have been almost completely ignored in fracture mechanics literature it seems probable that they will play a role in general formulations of "admissability criteria" for crack propagation, (cf. the theory of shock waves).

Continuum mechanical methods are prevailing in the mathematical theory of fracture (Liebowitz [19], Broek [20], Knott [21]). This is natural since

at least a major part of a fracturing body, away from the crack, is accurately described in a continuum model.

From a physical-phenomenological standpoint it may be argued that close to the crack-tip, in the so-called "process-region", basic continuum mechanical concepts like stress and strain loose their meaning. A number of microscopic non-continuum phenomena, appearing in the process-region, would call for new mathematical methods (Broberg [22]). However, no serious attempt to construct such a non-continuum theory of fracture has yet been made (to the best of my knowledge) and the difficulties in such a project seem to be quite substantial.

In a continuum theory we ignore the processes ahead of the crack-tip (the micro-mechanisms) and consider the body as a classical continuum all the way up to the crack-tip. However, the creation of new boundary surface during the fracture process will call for an extended continuum theory involving not only the bulk of the body but also its boundary surface. Furthermore we have to accept the introduction of certain singularities at the crack-tip.

Despite the fact that continuum methods are used by many working in the field of fracture mechanics there seems to be no general agreement on how to formulate the basic balance laws. There are even authors who claim that mechanical laws universally accepted, for instance the balance of energy, can not be valid in a fracturing body (see Liu [23] and the dispute between R.M. Christensen and L.N. McCartney in [24]).

The position taken in this thesis is that in any closed and consistent theory of macroscopic bodies, whether including fracture or not, the balance of mass, momentum, moment of momentum, energy and entropy should be postulated as necessary conditions. Arguments in favour of this opinion may be found in the critical study by Eftis & Liebowitz ([17]).

The first to formulate a rational basis for continuum fracture theories was H.C. Strifors in his thesis ([10]).

In paper I we discuss a model for a fracturing body that may be seen as the natural extension of the model presented in [10] in that we give a more detailed thermo-mechanical description of the boundary of the body. The paper is devoted to the general principles of balance alone. We give a detailed account for the kinematics of the bulk, the boundary surface and the crack-front (crack-tip). That nearly one half of the work is occupied with geometrical and kinematical discussions reflects the intricacy of these subjects when material separation is involved.



A smooth surface is introduced as the analytical representative of a crack. The boundary curve of this surface constitutes the instantaneous crack-front along which the material is separating. We discuss a fundamental re-kinematical relation connecting the motion of the crack-front in the so-called spatial and referential descriptions (Propositions 2.3 and 2.3'). A substantial part of the kinematical discussion is devoted to an analysis of so-called transport theorems associated with field quantities defined in the bulk and on the boundary. These transport theorems are of major importance in the discussion of the balance laws. Due to the possible singularities at the crack-front the well-known formula

$$\frac{d}{dt} \int_{B_t} \Phi dv = \int_{B_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x}) dv \quad (4)$$

will, in general, not be valid. Under specific assumptions we may replace (4) by

$$\frac{d}{dt} \int_{B_t} \Phi dv = \int_{B_t} (\Phi' + \Phi \operatorname{div} x') dv - \int_{\partial B_t} \Phi (x' - \dot{x}) \cdot \bar{n} da \quad (5)$$

where Φ' is a convected time derivative associated with the motion of the crack-front (Propositions 2.9 and 2.11). This convected crack-front time derivative has been more or less explicitly present in the fracture mechanics literature for some time. A formal definition is given in Strifors [11]. The introduction of the convected crack-front time derivative is motivated by the conjecture that Φ' will be less singular at the crack-front than $\dot{\Phi}$.

We postulate a global master balance law (in integral form) for a fracturing body. The equation is presented in an abstract setting leaving open the thermo-mechanical interpretations of the participating fields. This approach has the advantage of clearly revealing the structural properties of our equations.

The main result of the paper is a theorem establishing the equivalence of on the one hand the global master balance law and on the other hand a number of conditions (in differential form) representing the local balance at material points in the bulk and on the boundary and a so-called reduced master balance law which is satisfied identically if the body does not contain a crack (Theorems 3.1 and 3.2).

The form of the reduced master balance law motivates the introduction of a concept called the release rate of the balanced quantity. It may be

interpreted as a kind of draw, from the bulk to the crack-front, of the balanced quantity. It turns out that the release rate is equal to zero if formula (4) is valid and if the divergence theorem may be applied to the region occupied by the body and the field representing the bulk efflux of the balanced quantity (Proposition 4.3).

A central problem in the theory is to find effective ways of computing the release rate. By introducing some extra regularity assumptions we are able to prove a result that gives a fairly simple representation of the release rate as a limit, the so-called residual form (Theorem 4.1, Proposition 4.5). The residual form is used to prove a local version of the reduced master balance law representing the local balance at a material point situated on the crack-front (Theorem 4.2).

In the final chapter we present the balance laws for mass, momentum, moment of momentum, energy and entropy for a fracturing body modelled by a non-polar, single-phase continuum. We demonstrate a number of simplifications arising from suitable combinations of the balance laws and from the introduction of additional regularity assumptions.

In paper II we consider a two-dimensional body of hyperelastic material (hyperelastic bulk and hyperelastic boundary). The body is assumed to contain a straight-lined edge-crack. We formulate a boundary-initial-value problem for the fracturing body. This problem contains a number of conditions representing the reduced balance of momentum, moment of momentum and energy for the fracturing body (cf. paper I). We use the convected crack-front time derivative to prove an elegant reformulation of the reduced balance of energy originally presented in Rice [25] (Theorems 1 and 1a). We also discuss kinematical assumptions sufficient to secure that inertial effects play no part in the reduced balance of energy (quasi-static conditions) and we present a formula for the determination of the crack-tip speed in a material that is stable in the sense of Drucker ([26]).

A considerable amount of effort in fracture mechanics has been put into the formulation of so-called fracture-criteria. Many of these fracture-criteria are founded on energy balance considerations, sometimes supported by more doubtful speculations. In the technologically important case concerning the onset of fracture the best known criteria are based on the concepts: stress intensity factor (Irwin [5]) and J-integral (Rice [27], Broberg [28]).

Paper III is a contribution to the theoretical foundations of the J-integral method. We give a rigorous demonstration, based on rather weak

kinematical assumptions, of the J-integral fracture criterion in the linearized theory of elasticity.

In paper IV we state a boundary-value problem in linear elasto-statics for a two-dimensional body containing a crack. The model contains the possibility of non-zero tensions on the crack-surfaces. We also consider the possibility that the crack-surfaces may be in contact.

We prove uniqueness of the solution to the boundary-value problem within a class of deformations that produce moderate stress-singularities at the crack-tip.

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PAPER I

EQUATIONS OF BALANCE
IN
CONTINUUM FRACTURE MECHANICS

by

Per Lidström

ABSTRACT

In the present study we give a unified treatment of the balance laws in continuum fracture mechanics.

The model presented contains a detailed thermo-mechanical description of both the bulk and the boundary surface of the fracturing body.

We postulate a global master balance law for the fracturing body and derive equations representing the local balance at material points in the bulk and on the boundary. Special attention is given to the equation representing the local balance at a material point undergoing fracture.

INTRODUCTION AND SCOPE OF THE PAPER

Continuum mechanical methods are prevailing in the mathematical theory of fracture. This is natural since at least a major part of a fracturing body, away from the crack, is accurately described in a continuum model. From a physical-phenomenological standpoint it may be argued that close to the crack-tip, in the so-called "process-region", basic continuum mechanical concepts like stress and strain loose their meaning. The microscopic phenomena, appearing in the process-region, can not be modelled in a classical continuum theory. New mathematical methods are required.

In a continuum theory we ignore the micro-processes at the crack-tip and consider the whole body, including the process-region, as a classical continuum. This approach has the advantage of only using well-defined and familiar concepts from continuum mechanics. However, the creation of new boundary surface during the fracture process calls for an extended description in which the boundary is considered as a bidimensional continuum. It also gives rise to a number of intricate kinematical problems and we may have to accept the introduction of certain singularities at the crack-tip. Despite the flourishing literature on the subject, the continuum theory and its implications seem to be far from fully investigated. This is in particular the case for problems relating to the foundations of the theory. In this paper we discuss the equations of balance in continuum fracture mechanics. Our investigations are motivated by and to some extent based on the work of H.C. Strifors ([1,2]). We give a few words concerning the content of this paper.

In chapter 1 we present our scheme of notation and introduce some preliminary definitions.

Chapter 2 is devoted to geometry and kinematics. A smooth surface with moving boundary is introduced as the analytical representative of a crack. We present the basic concepts associated with the deformation and motion of material points in the bulk and on the boundary of the body. The parallel treatment of the bulk and the boundary will display several structural similarities. We discuss a fundamental kinematical relation associated with the motion of the crack-front (crack-tip) (Propositions 2.3 and 2.3'). In section 2.2 we discuss so-called transport theorems for field quantities defined in the bulk and on the boundary. In fracture mechanics the conditions at the crack-front are of primary interest. We demonstrate the existence of a useful system of coordinates associated with the so-called

tubular neighbourhood of the crack-front. We introduce and discuss a convected time derivative associated with the motion of the crack-front. This derivative is used in the discussion of the transport theorem for bulk quantities (Propositions 2.9 and 2.11). In section 2.4 we record some notions related to singular surfaces.

Chapter 3 is devoted to the balance laws. We postulate a global master balance law for a fracturing body. The main result in section 3.1 is a theorem demonstrating the equivalence of on the one hand the global master balance law and on the other hand a number of conditions (in differential form) representing the local balance at material points in the bulk and on the boundary and a so-called reduced master balance law which is satisfied identically if the body does not contain a crack (Theorems 2.1 and 2.2). In section 3.2 we deal with the relation between the referential and spatial descriptions.

Chapter 4 is concerned with the so-called release rate. The introduction of this concept is motivated by the structure of the reduced master balance law. It is important to find effective ways of computing the release rate. In section 4.1 we give a representation of the release rate in the form of a limit (Theorem 4.1). This so-called residual form of the release rate is used to prove a local version of the reduced master balance law representing the local balance at material points situated on the crack-front (Theorem 4.2).

In chapter 5 we present the balance laws for mass, momentum, moment of momentum, energy and entropy for a fracturing body modelled by a non-polar single-phase continuum. We discuss simplifications of the local balance at the crack-front that may be obtained if the body is assumed to undergo a so-called moderate fracture motion.

To make the paper reasonably self-contained we have appended some results from algebra and analysis used in our discussions.

1. BASIC NOTATIONS AND PRELIMINARY DEFINITIONS

In this section we give some information on the scheme of notation used in this paper. We also present a number of preliminary definitions. The paper makes systematic use of direct rather than component notation for representing vector and tensor quantities. The general scheme follows mainly that of Truesdell & Noll [3] and Truesdell [4].

Let R denote the set of real numbers: $a, b, c, \dots, q, r, s, \dots$. Let E denote a 3-dimensional Euclidean point space with translation space V . Points in E are denoted: $X, Y, Z, \dots, x, y, z, \dots$ and vectors in V are denoted: $\bar{a}, \bar{b}, \bar{c}, \dots, \bar{u}, \bar{v}, \bar{w}, \dots$. The scalar, vector and tensor products between two vectors $\bar{a}, \bar{b} \in V$ are denoted by $\bar{a} \cdot \bar{b}$, $\bar{a} \times \bar{b}$ and $\bar{a} \otimes \bar{b}$ respectively.¹ The absolute value ("length") of a vector $\bar{a}$ is denoted $|\bar{a}|$, where $|\bar{a}| = \sqrt{\bar{a} \cdot \bar{a}}$. Let U, W designate finite-dimensional inner product spaces and let

$L(U, W) \equiv$ the space of linear mappings from U into W .

$IL(U, W) \equiv \{ \underline{A} \in L(U, W) \mid \underline{A} \text{ is bijective} \}$

$\text{End}(U) \equiv L(U, U)$

$\text{Aut}(U) \equiv IL(U, U)$

If $\underline{A} \in IL(U, W)$ then the inverse of $\underline{A}$, denoted $\underline{A}^{-1}$, exists.² We have $\underline{A}^{-1} \in IL(W, U)$ and

$$\underline{A}^{-1} \underline{A} = \underline{1}_U, \quad \underline{A} \underline{A}^{-1} = \underline{1}_W \quad (1.1)$$

where $\underline{1}_U$ is the unit tensor on U . Given $\underline{A} \in L(U, W)$ we denote by $\underline{A}^T$ the transpose of $\underline{A}$, i.e. $\underline{A}^T \in L(W, U)$ and

$$\bar{w} \cdot \underline{A} \bar{u} = \bar{u} \cdot \underline{A}^T \bar{w} \quad (1.2)$$

for every $\bar{u} \in U$, $\bar{w} \in W$. (We use the same symbol " $\cdot$ " for the inner product on any inner product space. The underlying space will always be clear from the context). For $\underline{A} \in \text{End}(U)$ let $\text{tr} \underline{A}$ and $\det \underline{A}$ denote respectively the trace and determinant of $\underline{A}$. An inner product is defined on the vector space $L(U, W)$ by

¹ $(\bar{a} \otimes \bar{b}) \bar{v} = \bar{a} \bar{b} \cdot \bar{v}$, $\bar{v} \in V$. ² $\dim(U) = \dim(W)$

$$\underline{A} \cdot \underline{B} \equiv \text{tr}(\underline{A} \underline{B}^T) = \text{tr}(\underline{A}^T \underline{B}) \quad (1.3)$$

for all $\underline{A}, \underline{B} \in L(U, W)$. The absolute value $|\underline{A}|$ of $\underline{A} \in L(U, W)$ is defined by

$$|\underline{A}| \equiv \sqrt{\underline{A} \cdot \underline{A}} \quad (1.4)$$

Let U be a 3-dimensional inner product space (with vector product " $\times$ ") If $\underline{A} \in \text{Aut}(U)$ then we have the formula

$$(\underline{A} \bar{u}) \times (\underline{A} \bar{v}) = (\det \underline{A}) \underline{A}^{-T} (\bar{u} \times \bar{v}) \quad (1.5)$$

for all $\bar{u}, \bar{v} \in U$. Here $\underline{A}^{-T} \equiv (\underline{A}^{-1})^T$. We say that $\underline{A} \in \text{End}(U)$ is symmetric if $\underline{A}^T = \underline{A}$, skew-symmetric if $\underline{A}^T = -\underline{A}$ and positive definite if $\bar{u} \neq \bar{0}$ ($\bar{0}$ denotes the zero vector in U) implies $\bar{u} \cdot \underline{A} \bar{u} > 0$. Let

$$\text{Sym}(U) \equiv \{ \underline{A} \in \text{End}(U) \mid \underline{A} \text{ is symmetric} \}$$

$$\text{Sym}^+(U) \equiv \{ \underline{A} \in \text{Sym}(U) \mid \underline{A} \text{ is positive definite} \}$$

$$\text{Skew}(U) \equiv \{ \underline{A} \in \text{End}(U) \mid \underline{A} \text{ is skew symmetric} \}$$

A tensor $\underline{Q} \in L(U, W)$ is said to be orthogonal if

$$\underline{Q}^T \underline{Q} = \underline{1}_U, \quad \underline{Q} \underline{Q}^T = \underline{1}_W \quad (1.6)$$

Let

$$O(U, W) \equiv \{ \underline{Q} \in L(U, W) \mid \underline{Q} \text{ is orthogonal} \}$$

$$O(U) \equiv O(U, U)$$

Let W be a 3-dimensional inner product space (with vector product " $\times$ ") Let $\underline{A} \in L(U, W)$ and $\bar{a} \in W$ then the mapping $\bar{a} \times \underline{A} \in L(U, W)$ is defined by

$$(\bar{a} \times \underline{A}) \bar{u} \equiv \bar{a} \times (\underline{A} \bar{u}), \quad \bar{u} \in U \quad (1.7)$$

Let $\underline{A} \in \text{Skew}(W)$, then there is a unique $\text{ax}(\underline{A}) \in W$, called the axial vector of $\underline{A}$, satisfying

$$\underline{A} = ax(\underline{A}) \times \underline{1}_W \quad (1.8)$$

In continuum physics the mathematical model of a body is a manifold characterized by its configurations in E . In this paper the body is identified by a fixed global reference configuration B , a regular region in E . A typical point $X \in B$ is called a material point. A deformation of the body, from the reference configuration, is defined by a mapping $\chi : B \rightarrow E$

$$x = \chi(X), \quad x \in E$$

A motion of the body is a family of deformations

$$x = \chi(X, t)$$

(smoothly) parametrized by time $t \in T$. Material velocity is defined by

$$\dot{x} = \frac{\partial}{\partial t} \chi(X, t)$$

The various fields entering continuum physics, such as material velocity, stress, temperature etc., can be represented equally well as functions of (X, t) (the referential description) or (x, t) (the spatial description). Both are useful, but introduce a cumbersome notational problem. For instance, in the case of temperature, logically we need three different symbols: one for the value, one for the referential description and one for the spatial description

$$\theta = \Theta(X, t) = \Xi(x, t)$$

However, one traditionally overlooks this notational precision, using the single symbol θ to denote the value as well as the functions Θ and Ξ . To avoid confusion in the representation of derivatives one introduces the differential operators

$$\text{Grad } \theta \equiv \frac{\partial}{\partial X} \Theta \quad \text{referential gradient}$$

$$\text{grad } \theta \equiv \frac{\partial}{\partial x} \Xi \quad \text{spatial gradient}$$

(1.9)_{1,2}

$$\dot{\theta} \equiv \frac{\partial}{\partial t} \theta \quad \text{referential (material) time derivative}$$

$$\partial_t \theta \equiv \frac{\partial}{\partial t} \theta \quad \text{spatial time derivative} \quad (1.9)_{3,4}$$

We have the following obvious relations

$$\begin{aligned} \dot{\theta} &= \partial_t \theta + (\text{grad } \theta) \cdot \dot{x} \\ \text{Grad } \theta &= \left(\frac{\partial}{\partial X} \chi \right)^T \text{grad } \theta \end{aligned} \quad (1.10)$$

The notations Div and div shall stand for the divergences formed from Grad and grad respectively.

Let S be a surface in B (a material surface) and let θ be a field defined on S . The operations: $\text{Grad}_S \theta$, $\text{grad}_S \theta$, $\text{Div}_S \theta$ and $\text{div}_S \theta$ are defined in a manner analogous to (1.9) using the surface gradients on S and $\chi(S)$.

2. GEOMETRY AND KINEMATICS

It is a major assumption in continuum mechanics that functions (and their derivatives) representing the behaviour of real bodies (motion, stress, temperature etc.) possess the quality of continuity. According to the definition of continuous motion it is described by a mapping

$$x = X(X,t), \quad X \in B, \quad t \in T$$

which together with its inverse

$$X = X^{-1}(x,t), \quad x \in X(B,t), \quad t \in T$$

must be single-valued and differentiable a suitable number of times. (cf. Truesdell & Toupin [5], p. 325.) These strong requirements do, when globally applied to the body considered, rule out a number of important phenomena in physics and technology. Among these are shocks, the action of concentrated loads, cavitation, fracture and its inverse welding. The treatment of these problems within the framework of continuum mechanics calls for an extended class of mappings exhibiting various types of singular behaviour. For instance, in the theory modelling wave-propagation in materials (cf. Coleman et. al. [6].) one is dealing with a class of mappings (fields) that are smooth everywhere, except on a family of smooth propagating surfaces (singular surfaces) across which they experience jump-discontinuities.

In the process of fracture (tearing) a singular line F , the crack-front, is propagating into the body, splitting each material point X in its progress, into two material points X^+ and X^- , joining two separate, newly created surfaces S^+ and S^- , the crack-surfaces.

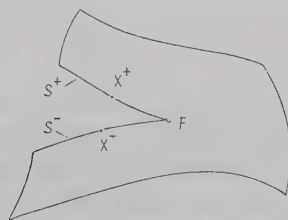


Figure 2.1

This creation of new boundaries, being conceptually and mathematically very intricate, is the characteristic quality of fracture kinematics. In a continuum mechanical description of this process we will generally have to admit fields that are essentially singular (unbounded) at the crack-front and this fact introduces a number of non-trivial analytical problems.

It is the main purpose of this chapter to equip ourselves with a class of mappings (deformations) general enough to be able to simulate "tearing behaviour" and suitable for our later discussion on the equations of balance.

2.1 Fracture deformation

A body is identified by a reference configuration B , a fixed regular region in Euclidean space E . The orientation on ∂B (the boundary of B) is denoted $\bar{n}$ (the outward unit normal on ∂B). Let S be a smooth surface with boundary in E and let $\bar{m}$ be an orientation on S . Put

$$\underline{S} \equiv S \cap \bar{B}, \quad \underline{F} \equiv \partial S \cap \bar{B}$$

The crack in the body B , i.e. the set of material points in B that have been splitted in the fracture process, is modelled by the surface $\underline{S} \setminus \underline{F}$. We call $\underline{S}$ the crack-surface in the referential description. The space curve $\underline{F}$ is referred to as the crack-front in the referential description. We assume that $\underline{F} \neq \emptyset$ and that $\underline{S}$ is a regular subregion of S . If $\underline{F} = \partial \underline{S}$ then the crack is called an internal crack, otherwise it is called an edge-crack. The geometry in the reference configuration is illustrated in the figure below.

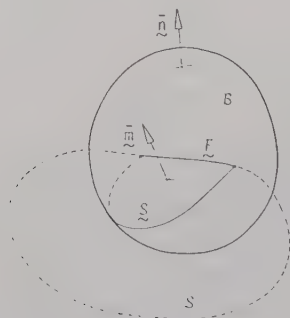


Figure 2.2

Remark: In an axiomatic presentation of a continuum mechanical theory the determination of the appropriate collection of sets in E for the representation of bodies (a universe) is of fundamental importance (see Truesdell [5], chap. I). Here we take, somewhat naively, the regular regions defined in Appendix A.8 as a representative for the family of bodies in the referential description. This class of sets fulfills some of our practical requirements. The boundaries may have corners and edges and they have a well-defined exterior normal almost everywhere. Furthermore, the divergence theorem applies to these regions (cf. Appendix A.8). Inconsistencies that may appear in an axiomatic structure based on this family are ignored in this paper.

Let $\chi: \bar{B} \setminus \underline{S} \rightarrow E$ be a smooth mapping. To this we associate the two mappings

$$\bar{\chi}^+: \underline{S} \rightarrow E, \quad \bar{\chi}^-: \underline{S} \rightarrow E$$

defined by

$$\bar{\chi}^+(Z) \equiv \lim_{X \rightarrow Z+} \chi(X), \quad \bar{\chi}^-(Z) \equiv \lim_{X \rightarrow Z-} \chi(X), \quad Z \in \underline{S} \setminus \underline{F} \quad (2.1)$$

$$\bar{\chi}^+(Z) = \bar{\chi}^-(Z) \equiv \lim_{\substack{X \rightarrow Z \\ X \in \bar{B} \setminus \underline{S}}} \chi(X), \quad Z \in \underline{F} \quad (2.2)$$

where the limits in (2.1) should be understood according to the following construction: Given $Z \in \underline{S} \setminus \underline{F}$ then there is an open ball $B(Z, r)$ (centered at Z and with radius $r > 0$) and a smooth mapping $f: B \rightarrow \mathbb{R}$ such that $\underline{S} \cap B = \{X \in B \mid f(X) = 0\}$ and $\bar{m}(Z) \cdot \nabla f(Z) > 0$ for $Z \in \underline{S} \cap B$. We define

$$\lim_{X \rightarrow Z+} \chi(X) \equiv \lim_{\substack{X \rightarrow Z \\ X \in B, f(X) > 0}} \chi(X), \quad \lim_{X \rightarrow Z-} \chi(X) \equiv \lim_{\substack{X \rightarrow Z \\ X \in B, f(X) < 0}} \chi(X) \quad (2.3)$$

The limits $\bar{\chi}^\pm(Z)$, $Z \in \underline{S} \setminus \underline{F}$ are thus obtained by approaching Z on paths lying entirely on the positive (+) respectively negative (-) side of the crack-surface. In the same manner we define: $(\nabla \chi)^+: \underline{S} \rightarrow \text{End}(V)$,¹ $(\nabla \chi)^-: \underline{S} \rightarrow \text{End}(V)$ and limits of higher order derivatives $(\nabla^2 \chi)^\pm, \dots$. We will write $(\phi)^\pm$ for the limits defined as in (2.1) for any field

¹ $\nabla \equiv d/dX$

$$\phi : \bar{S} \setminus \underline{S} \rightarrow \mathbb{R}, \nu, \text{End}(V), \dots$$

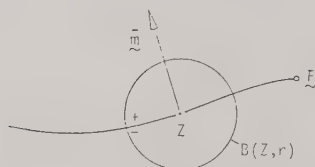


Figure 2.3

A diffeomorphism

$$\chi : \bar{S} \setminus \underline{S} \rightarrow \chi(\bar{S} \setminus \underline{S})$$

is called a fracture-deformation if

K1: $\chi(\bar{S} \setminus \underline{S})$ is a bounded set in E .

K2: the mappings $\bar{\chi}^+$, $\bar{\chi}^-$, defined above, exist, $S^+ \equiv \bar{\chi}^+(\underline{S})$, $S^- \equiv \bar{\chi}^-(\underline{S})$ are regular surfaces (with boundary) and the mappings $\chi^+ : \underline{S} \rightarrow S^+$, $\chi^- : \underline{S} \rightarrow S^-$ are diffeomorphisms. (Here $\chi^\pm$ are identical to $\bar{\chi}^\pm$ when the codomains of the latter functions are restricted to their ranges).

K3: the limits $(\nabla \chi)^\pm$, $(\nabla^2 \chi)^\pm$, ..., defined in accordance with (2.1), exist for all $Z \in \underline{S} \setminus \underline{F}$.

By this definition the crack-front in the spatial description

$$F \equiv \chi^+(F_-) = \chi^-(F_-)$$

is a well-defined smooth space curve. The body is, in the spatial description, represented by the bounded set $\chi(\bar{S} \setminus \underline{S}) \cup S^+ \cup S^-$. Note that this need not be a regular region since the regularity properties of the set, at the crack-front, are unknown. The boundary may have a "cusp" at the crack-front.

The orientation of the boundary surface $\partial(\chi(\bar{S} \setminus \underline{S}) \cup S^+ \cup S^-)$ is given by

the unit normal vector fields $\tilde{n}$ and $\tilde{m}$. See the illustration below.

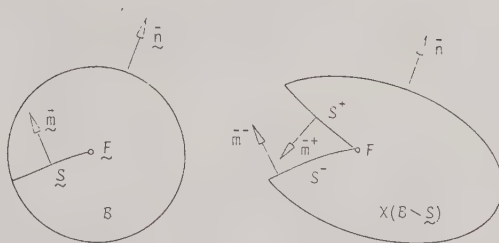


Figure 2.4

The material points of the fracturing body are divided into the following natural categories:

- $X \in \overset{\circ}{B} \setminus \underset{\sim}{S}$ bulk material point
- $X \in \underset{\sim}{S}$ crack-surface material point
- $X \in \underset{\sim}{F}$ crack-front material point
- $X \in \partial B$ boundary material point

The primary task of this paper is to establish local balance laws at the material points of a fracturing body. The structure of these balance laws will strongly depend on whether we are considering a bulk material point, a crack-surface material point, a crack-front material point or a boundary material point.

Remark 1: In our definition of a fracture-deformation we have left open the behaviour of the derivatives $\nabla \chi$, $\nabla^2 \chi, \dots$ as $\underset{\sim}{F}$ is approached from $B \setminus \underset{\sim}{S}$. This is of course motivated by the singular character of certain known (asymptotic) solutions to the equations of the linear theory of elasticity giving stresses and strains in crack-like configurations (see Sneddon & Lowengrub [7]).

Remark 2: We do not exclude the possibility that parts of the crack-surfaces S^+ and S^- are in contact. Note that we may even have $\bar{\chi}^+(X) = \bar{\chi}^-(X)$ on parts of $\underset{\sim}{S} \setminus \underset{\sim}{F}$.

The definition of a fracture-deformation contains the assumption that the mappings $\chi^\pm: \underline{S} \rightarrow S^\pm$ are smooth. We may use Hadamard's lemma (see Appendix A.7) to establish relations between derivatives of $\chi^\pm$ and limits of derivatives of χ at points away from the crack-front.

The deformation gradients:

$$\underline{F} \equiv \nabla \chi, \quad \underline{F}^+ \equiv \nabla_{\underline{S}} \chi^+ \quad \text{and} \quad \underline{F}^- \equiv \nabla_{\underline{S}} \chi^-$$

are related by

Proposition 2.1

$$\underline{F}^+(\chi^\pm(X)) \underline{F}^\pm(X) = (\underline{F})^\pm(X) \underline{I}(X), \quad X \in \underline{S} \setminus \underline{F} \quad (2.4)$$

where $\underline{I}(X)$ and $\underline{I}^\pm(X)$ are the inclusion mappings: $T_X \rightarrow V$ and $T_X^\pm \rightarrow V$. Here T_X and $T_X^\pm$ denote the tangent spaces of the surfaces $\underline{S}$ and $S^\pm$ at points X and x respectively. The surface gradient is denoted by the symbol $\nabla_{\underline{S}}$. For definitions of the inclusion mapping and the surface gradient see Appendix A.6.

Proof: The relation (2.4) is a direct consequence of Hadamard's lemma. $\square$

Remark: On account of the possible discontinuity of χ on $\underline{S} \setminus \underline{F}$ (i.e. $\chi^+ \neq \chi^-$) $\underline{F}^+(X)$ is in general unrelated to $\underline{F}^-(X)$. However, if $\chi^+ = \chi^-$ in a neighbourhood of $X \in \underline{S} \setminus \underline{F}$ then we have $\underline{F}^+(X) = \underline{F}^-(X)$ and $[\underline{F}](X) = (\underline{F})^+(X) - (\underline{F})^-(X) = [(\nabla \chi)(X) \underline{m}(X)] \otimes \underline{m}(X)$.¹

It is an immediate consequence of our definition of a fracture-deformation that

$$\underline{F}(X) \in \text{Aut}(V), \quad X \in \bar{B} \setminus \underline{S}$$

$$(\underline{F})^\pm(X) \in \text{Aut}(V), \quad X \in \underline{S} \setminus \underline{F}$$

$$\underline{F}^\pm(X) \in \text{IL}(T_X, T_X^\pm(X)), \quad X \in \underline{S}$$

and we have the polar decompositions²

$$\underline{F} = \underline{R} \underline{U}, \quad \underline{F}^\pm = \underline{R}^\pm \underline{U}^\pm \quad (2.5)$$

with

¹ cf. Truesdell & Toupin [5], p. 494. ² see Appendix A.2

$$\underline{U}(X) \in \text{Sym}^+(V), \quad X \in \bar{B} \setminus \bar{F}^{-1}$$

$$\underline{R}(X) \in O(V), \quad X \in \bar{B} \setminus \bar{F}^{-1}$$

$$\underline{U}^\pm(X) \in \text{Sym}^+(\tau_X), \quad X \in \underline{S}$$

$$\underline{R}^\pm(X) \in O(\tau_X, \tau_X^\pm(X)), \quad X \in S$$

The tensors $\underline{U}$, $\underline{U}^\pm$ are called the right stretch tensors and $\underline{R}$, $\underline{R}^\pm$ are called rotation tensors. Put $\bar{\underline{F}}^\pm \equiv \nabla_{\bar{X}} \bar{\chi}^\pm$ then

$$\bar{\underline{F}}^\pm(X) \in L(\tau_X, V), \quad X \in \underline{S}$$

and we have the simple relation

$$\bar{\underline{F}}^\pm(X) = \underline{I}^\pm(\bar{\chi}^\pm(X)) \underline{F}^\pm(X), \quad (2.6)$$

Put $\bar{\underline{R}}^\pm \equiv \underline{I}^\pm \underline{R}^\pm$ then

$$\bar{\underline{R}}^\pm(X) \in L(\tau_X, V), \quad X \in \underline{S}$$

$$(\bar{\underline{R}}^\pm)^T \bar{\underline{R}}^\pm = \underline{1}_T, \quad \bar{\underline{R}}^\pm (\bar{\underline{R}}^\pm)^T = \underline{1}_V - \bar{\underline{m}}^\pm \otimes \bar{\underline{m}}^\pm \quad (2.7)$$

$$\bar{\underline{F}}^\pm = \bar{\underline{R}}^\pm \underline{U}^\pm$$

Here $\underline{1}_T$ and $\underline{1}_V$ denote the unit tensors on T and V respectively. Note that

$$\bar{\underline{F}}^\pm = \underline{P}^\pm \bar{\underline{F}}^\pm \quad (2.8)$$

where $\underline{P}^\pm(x)$ is the perpendicular projection: $V \rightarrow T_x^\pm$.² This follows from (2.6) and the basic relations

$$\underline{P}^\pm \underline{I}^\pm = \underline{1}_T^\pm, \quad \underline{I}^\pm \underline{P}^\pm = \underline{1}_V - \bar{\underline{m}}^\pm \otimes \bar{\underline{m}}^\pm \quad (2.9)$$

where $\underline{1}_T^\pm$ is the unit tensor on $T^\pm$.

The right Cauchy-Green tensors $\underline{C}$ and $\underline{C}^\pm$ are defined by

$$\underline{C} \equiv \underline{U}^2, \quad \underline{C}^\pm \equiv (\underline{U}^\pm)^2 \quad (2.10)$$

¹ Limits $(\underline{U})^\pm$, $(\underline{R})^\pm$ are included. ² See Appendix A.6

Note that

$$\underline{C} = \underline{F}^T \underline{F}, \quad \underline{C}^\pm = (\underline{F}^\pm)^T \underline{F}^\pm = (\underline{\tilde{F}}^\pm)^T \underline{\tilde{F}}^\pm \quad (2.11)$$

The volume ratio J associated with the local deformation $\underline{F}$ is defined by

$$J \equiv \det \underline{U} \quad (2.12)$$

and $0 < J < \infty$ for $X \in \bar{B} \setminus \underline{F}$.

Area ratios $A^\pm$ associated with the local deformations $\underline{F}^\pm$ are defined by

$$A^\pm \equiv \det \underline{U}^\pm \quad (2.13)$$

and $0 < A^\pm < \infty$ for $X \in \underline{S}$.

Remark: Note that in contrast to the volume ratio, which may be singular (unbounded) at the crack-front, the area-ratios are well-defined on the crack-front.

We now turn to the discussion of some important kinematical quantities.

Let T be a bounded open time-interval ($T \equiv]t_1, t_2[, t_1, t_2 \in \mathbb{R}$).

Consider a family of fracture-deformations

$$X(\cdot, t) : \bar{B} \setminus \underline{S}_t \rightarrow B_t \equiv X(\bar{B} \setminus \underline{S}_t, t), \quad t \in T$$

having the properties:

K4: $\underline{S}_t \equiv S_t \cap \bar{B}$ where S_t , $t \in T$ is a moving surface-with-boundary in E with zero normal velocity. (The reader is referred to Appendix A.6 for definitions and details on moving surfaces).

K5: If B^- is an open subset of B and T^- an open subset of T satisfying $B^- \cap \underline{S}_t = \emptyset$ for all $t \in T^-$ then the mapping

$$B^- \times T^- \ni (X, t) \leadsto X(X, t)$$

is smooth (or at least C^2).

K6: If T^- is an open subset of T and S^- an open subset of $\underline{S}_t \setminus \underline{F}_t$ for

all $t \in T^-$ then the mappings

$$S^- \times T^- \ni (X, t) \mapsto \bar{X}^\pm(X, t)$$

are smooth (or at least C^2)

We have the simple observation

Lemma 2.1 a) For any time $t \in T$ and any bulk material point $X \in B \setminus \underline{S}_t$ there is an open time-interval $T^- \ni t$ and an open set $B^- \ni X$ such that $B^- \subset B \setminus \underline{S}_s$ for all $s \in T^-$.

b) For any time $t \in T$ and any crack-surface material point $X \in \underline{S}_t \setminus \underline{F}_t$ there is an open time-interval $T^- \ni t$ and an open set $S^- \subset \underline{S}_t \setminus \underline{F}_t$ such that $X \in S^- \subset \underline{S}_s \setminus \underline{F}_s$ for all $s \in T^-$.

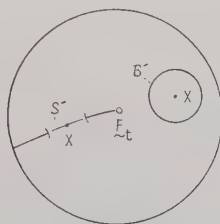


Figure 2.5

Proof: Omitted. $\square$

Lemma 2.1 and the assumptions made in K5 and K6 permit us to define the material velocity and acceleration:

$$\dot{X}(X, t) \equiv \frac{\partial}{\partial t} X(X, t) \quad X \in B \setminus \underline{S}_t$$

$$\ddot{X}(X, t) \equiv \frac{\partial^2}{\partial t^2} X(X, t)$$

$$\dot{\bar{X}}^\pm(X, t) \equiv \frac{\partial}{\partial t} \bar{X}^\pm(X, t)$$

$$\ddot{\bar{X}}^\pm(X, t) \equiv \frac{\partial^2}{\partial t^2} \bar{X}^\pm(X, t) \quad X \in \underline{S}_t \setminus \underline{F}_t$$

and we assume that

K7: The limits $(\dot{x})^\pm, (\ddot{x})^\pm, (\dot{F})^\pm, \dots$, in the sense of (2.1), exist.

The following compatibility conditions, representing the fact that crack-surfaces do not slip relative to the bulk material, are satisfied due to Hadamard's lemma (applied in the product space $E \times \mathbb{R}$).

Proposition 2.2

$$\dot{x}^\pm(X, t) = (\dot{x})^\pm(X, t)$$

$$\ddot{x}^\pm(X, t) = (\ddot{x})^\pm(X, t) \quad X \in \mathcal{S}_t \setminus \mathcal{F}_t$$

Remark: Our analysis is , except for the possible singularity at the crack-front, based on classical smoothness assumptions. The fields

$$\dot{x}(\cdot, t), \ddot{x}(\cdot, t), \dot{F}(\cdot, t), \dots$$

are continuous on $B \setminus \mathcal{S}_t$ and the fields

$$\dot{x}^\pm(\cdot, t), \ddot{x}^\pm(\cdot, t), \dot{F}^\pm(\cdot, t), \dots$$

are continuous on $\mathcal{S}_t \setminus \mathcal{F}_t$. However, in section 2.4 we will generalize our kinematical assumptions somewhat to include the possibility of propagating singular surfaces in B (shock waves, acceleration waves etc.).

Tangential velocity and acceleration fields on the crack-surfaces are defined by

$$\begin{aligned} \dot{x}_{||}^\pm &\equiv \underline{p}^\pm \dot{x}^\pm \\ \ddot{x}_{||}^\pm &\equiv \underline{p}^\pm \ddot{x}^\pm \end{aligned} \quad X \in \mathcal{S}_t \setminus \mathcal{F}_t \quad (2.14)$$

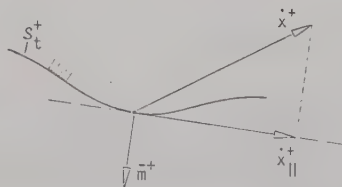


Figure 2.6

The spatial velocity gradients $\underline{G}$, $\underline{G}^\pm$ and $\underline{PG}^\pm$ are defined by

$$\begin{aligned}\underline{G} &= \text{grad } \dot{\underline{x}} \\ \underline{G}^\pm &= \text{grad}_\delta \dot{\underline{x}}^\pm \\ \underline{PG}^\pm &= \underline{P}^\pm \underline{G}^\pm\end{aligned}\quad (2.15)$$

and we have the following decompositions

$$\begin{aligned}\underline{G} &= \underline{D} + \underline{W} \\ \underline{PG}^\pm &= \underline{D}^\pm + \underline{W}^\pm\end{aligned}\quad (2.16)$$

where $\underline{D}$ and $\underline{D}^\pm$ are called stretchings

$$\underline{D}(X, t) \in \text{Sym}(V), \quad X \in B \setminus \underline{S}_t, \quad \underline{D}^\pm(X, t) \in \text{Sym}(T_{X^\pm}^\pm(X, t)), \quad X \in \underline{S}_t \setminus \underline{F}_t$$

and $\underline{W}$, $\underline{W}^\pm$ are called spins

$$\underline{W}(X, t) \in \text{Skew}(V), \quad X \in B \setminus \underline{S}_t, \quad \underline{W}^\pm(X, t) \in \text{Skew}(T_{X^\pm}^\pm(X, t)), \quad X \in \underline{S}_t \setminus \underline{F}_t$$

We record a number of useful kinematical relations

Lemma 2.2

$$\begin{aligned}\underline{G} &= \dot{\underline{F}} \underline{F}^{-1} \\ \underline{D} &= \frac{1}{2} \underline{R} (\dot{\underline{U}} \underline{U}^{-1} + \underline{U}^{-1} \dot{\underline{U}}) \underline{R}^T \\ \underline{W} &= \dot{\underline{R}} \underline{R}^T + \frac{1}{2} \underline{R} (\dot{\underline{U}} \underline{U}^{-1} - \underline{U}^{-1} \dot{\underline{U}}) \underline{R}^T \\ \underline{G}^\pm &= \dot{\underline{F}}^\pm (\underline{F}^\pm)^{-1} \\ \underline{D}^\pm &= \frac{1}{2} \underline{R}^\pm (\dot{\underline{U}}^\pm (\underline{U}^\pm)^{-1} + (\underline{U}^\pm)^{-1} \dot{\underline{U}}^\pm) (\underline{R}^\pm)^T \\ \underline{W}^\pm &= \underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T + \frac{1}{2} \underline{R}^\pm (\dot{\underline{U}}^\pm (\underline{U}^\pm)^{-1} - (\underline{U}^\pm)^{-1} \dot{\underline{U}}^\pm) (\underline{R}^\pm)^T\end{aligned}\quad (2.17)$$

Proof: We have (by using the chain rule and assumption K6)

$$\underline{G}^\pm = \text{grad}_\delta \dot{\underline{x}}^\pm = \text{Grad}_\delta \left(\frac{\partial}{\partial t} \bar{\underline{x}}^\pm \right) (\underline{F}^\pm)^{-1} = \frac{\partial}{\partial t} (\text{Grad}_\delta \bar{\underline{x}}^\pm) (\underline{F}^\pm)^{-1} = \dot{\underline{F}}^\pm (\underline{F}^\pm)^{-1}$$

and this proves (2.17)_L. Since $\dot{\underline{F}}^\pm = \dot{\underline{R}}^\pm \underline{U}^\pm + \underline{\dot{R}}^\pm \underline{\dot{U}}^\pm$ we get

$$\underline{P}\underline{G}^\pm = \underline{P}^\pm \dot{\underline{F}}^\pm (\underline{F}^\pm)^{-1} = \underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T + \underline{R}^\pm \underline{\dot{U}}^\pm (\underline{U}^\pm)^{-1} (\underline{R}^\pm)^T$$

and this can be written in the form

$$\begin{aligned} \underline{P}\underline{G}^\pm = \underline{D}^\pm + \underline{W}^\pm &= \underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T + \frac{1}{2} \underline{R}^\pm (\underline{\dot{U}}^\pm (\underline{U}^\pm)^{-1} - (\underline{U}^\pm)^{-1} \underline{\dot{U}}^\pm) (\underline{R}^\pm)^T + \\ &\quad \frac{1}{2} \underline{R}^\pm (\underline{\dot{U}}^\pm (\underline{U}^\pm)^{-1} + (\underline{U}^\pm)^{-1} \underline{\dot{U}}^\pm) (\underline{R}^\pm)^T \end{aligned}$$

By taking time derivatives on the relation $(\underline{\dot{R}}^\pm)^T \underline{R}^\pm = \underline{1}_T$ we get

$$\underline{R}^\pm (\underline{\dot{R}}^\pm)^T \underline{1}^\pm = - \underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T$$

Consequently

$$(\underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T)^T = - \underline{P}^\pm \dot{\underline{R}}^\pm (\underline{R}^\pm)^T$$

and the uniqueness of the additive decomposition of a tensor into symmetric and skew parts now proves (2.17)_{5,6}. The proof of (2.17)_{1,2,3} runs along the same lines (cf. Truesdell [4]). $\square$

Remark: Note that since $\underline{F}^\pm(X,t) \in \text{IL}(\tau_X, T_{X^\pm}^\pm(X,t))$ and $\underline{R}^\pm(X,t) \in O(\tau_X, T_{X^\pm}^\pm(X,t))$, $X \in \underline{S}_t$ the referential time derivative of $\underline{F}^\pm$ and $\underline{R}^\pm$ (i.e. what would be denoted $\dot{\underline{F}}^\pm$ and $\dot{\underline{R}}^\pm$) are not well-defined objects. The tensors $\dot{\underline{F}}^\pm(X,t)$, $\dot{\underline{R}}^\pm(X,t) \in L(\tau_X, V)$, $X \in \underline{S}_t \setminus \underline{F}_t$, used in formulas (2.17) are however perfectly well-defined.

We introduce the Weingarten mapping $\underline{L}$ associated with the crack-surface in the referential description $\underline{S}_t$

$$\underline{L} \equiv - \underline{P} \text{Grad}_\beta \bar{\underline{m}}$$

$(\underline{L}(X,t) \in \text{Sym}(\tau_X), X \in \underline{S}_t)$ and the mean curvature of $\underline{S}_t$

$$M \equiv \frac{1}{2} \text{tr } \underline{L}$$

Here $\underline{P}(X,t)$ denotes the perpendicular projection: $V \rightarrow \tau_X$, $X \in \underline{S}_t$.

Referential derivatives of the volume-ratio and the area-ratios are given by

Lemma 2.3

$$\dot{J} = J \operatorname{div} \dot{X}$$

$$\operatorname{Grad} J = J \operatorname{div} \underline{F}^T \quad (2.18)$$

$$\dot{A}^\pm = A^\pm \operatorname{div}_\delta \dot{X}^\pm$$

$$\operatorname{Grad}_\delta A^\pm = A^\pm (\operatorname{div}_\delta (\underline{F}^\pm)^T - 2 M \bar{m})$$

Proof: The identities $(2.18)_{1,3}$ are demonstrated by using the formula for differentiating a determinant (cf. Appendix A.3) together with $(2.17)_{2,5}$ in Lemma 2.2. Note that

$$\begin{aligned} \operatorname{div} \dot{X} &\equiv \operatorname{tr} \underline{G} = \operatorname{tr} \underline{D} = \dot{\underline{F}} \cdot \underline{F}^{-T} \\ \operatorname{div}_\delta \dot{X}^\pm &\equiv \operatorname{tr} \underline{P} \underline{G}^\pm = \operatorname{tr} \underline{D}^\pm = \dot{\underline{F}}^\pm \cdot \underline{I}^\pm (\underline{F}^\pm)^{-T} \end{aligned} \quad (2.19)$$

For the proof of $(2.18)_{2,4}$ we invoke the Piola-identities presented in section 3.2. A straightforward application of (3.28) and (3.40) gives

$$\begin{aligned} \operatorname{Div}(J \underline{1}_V) &= J \operatorname{div} \underline{F}^T \\ \operatorname{Div}_\delta (A^\pm \underline{1}_T) &= A^\pm \operatorname{div}_\delta (\underline{F}^\pm)^T \end{aligned} \quad (2.20)$$

and from the rules for calculating divergences we get

$$\begin{aligned} \operatorname{Div}(J \underline{1}_V) &= J \operatorname{Div} \underline{1}_V + \underline{1}_V \operatorname{Grad} J \\ \operatorname{Div}_\delta (A^\pm \underline{1}_T) &= A^\pm \operatorname{Div}_\delta \underline{1}_T + \underline{1}_T \operatorname{Grad}_\delta A^\pm \end{aligned} \quad (2.21)$$

But

$$\begin{aligned} \operatorname{Div} \underline{1}_V &= \bar{0} \\ \operatorname{Div}_\delta \underline{1}_T &= 2 M \bar{m} \end{aligned} \quad (2.22)$$

Combining (2.20) - (2.22) proves $(2.18)_{2,4}$. $\square$

The crack-surfaces, in the spatial description at time t , are represented by

$$S_t^\pm = \chi^\pm(\tilde{S}_t, t)$$

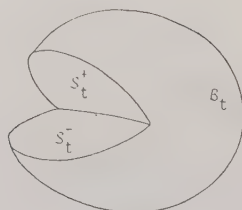


Figure 2.7

Weingarten mappings $\underline{L}^\pm$ associated with the surfaces $S_t^\pm$ are defined by

$$\underline{L}^\pm \equiv - \underline{P}^\pm \text{grad}_\delta \bar{m}^\pm \quad (2.23)$$

$(\underline{L}^\pm(X, t) \in \text{Sym}(\tau_{\tilde{X}}^\pm(X, t)), X \in \tilde{S}_t)$ and the mean curvatures by

$$M^\pm \equiv \frac{1}{2} \text{tr } \underline{L}^\pm \quad (2.24)$$

The tangential velocity gradients $\underline{PG}_{||}^\pm$ are defined by

$$\underline{PG}_{||}^\pm \equiv - \underline{P}^\pm \text{grad}_\delta \dot{\chi}^\pm \quad (2.25)$$

and their relation to $\underline{PG}^\pm$ is shown in

Lemma 2.4

$$\begin{aligned} \underline{PG}_{||}^\pm &= \underline{PG}^\pm + \underline{L}^\pm(\dot{\chi}^\pm \cdot \bar{m}^\pm) \\ \text{div}_\delta \dot{\chi}_{||}^\pm &= \text{div}_\delta \dot{\chi}^\pm + 2 M^\pm(\dot{\chi}^\pm \cdot \bar{m}^\pm) \end{aligned} \quad (2.26)$$

Proof:

$$\underline{PG}^\pm = \underline{P}^\pm \text{grad}_\delta \dot{\chi}^\pm = \underline{P}^\pm \text{grad}_\delta \dot{\chi}_{||}^\pm + \underline{P}^\pm \text{grad}_\delta (\bar{m}^\pm(\dot{\chi}^\pm \cdot \bar{m}^\pm)) = \underline{PG}_{||}^\pm +$$

$$+ (\dot{\bar{x}}^\pm \cdot \bar{m}^\pm) \underline{p}^\pm \operatorname{grad}_\delta \bar{m}^\pm + \underline{p}^\pm \bar{m}^\pm \otimes \operatorname{grad}_\delta (\dot{\bar{x}}^\pm \cdot \bar{m}^\pm)$$

But $\underline{L}^\pm = -\underline{p}^\pm \operatorname{grad}_\delta \bar{m}^\pm$ and $\underline{p}^\pm \bar{m}^\pm = \bar{0}$ and this proves (2.26)₁. By taking traces on (2.26)₁ we get (2.26)₂. $\square$

We have the decomposition

$$\underline{pG}_{||}^\pm = \underline{D}_{||}^\pm + \underline{W}^\pm \quad (2.27)$$

where

$$\underline{D}_{||}^\pm \equiv \underline{D}^\pm + \underline{L}^\pm (\dot{\bar{x}}^\pm \cdot \bar{m}^\pm) \quad (2.28)$$

Next we discuss some kinematical concepts relating to the propagation of the crack-front. Let

$$\tilde{\zeta} = \tilde{\zeta}_t(X), \quad X \in \tilde{F}_t$$

denote the normal-velocity of the moving curve $\tilde{F}_t$, $t \in T$. We call $\tilde{\zeta}$ the crack-front velocity in the referential description.

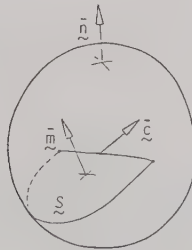


Figure 2.8

Put $\tilde{\zeta} = \tilde{v} \tilde{\zeta}$ where $\tilde{v} = \tilde{v}_t(X) \in T_X$ is the outward pointing (relative to $\tilde{S}_t$) unit normal on $\tilde{F}_t$. See the illustration above. We call $\tilde{\zeta} = \tilde{\zeta}_t(X)$ the crack-front speed in the referential description. Let

$$\bar{c} = \bar{c}_t(x), \quad x \in F_t$$

denote the normal-velocity of F_t , that is the crack-front velocity in

the spatial description. The existence of $\bar{c}$ is however not guaranteed by our assumptions so far. For, if $\hat{X}$ is a parametrization of the crack-front in the referential description, i.e.

$$\tilde{F}_t = \{X \in E \mid X = \hat{X}(q,t), \quad q \in [a,b]\}, \quad t \in T$$

then, in the spatial description, we have

$$F_t = \{x \in E \mid x = \hat{x}(q,t), \quad q \in [a,b]\}, \quad t \in T$$

where $\hat{x}(q,t) = \bar{x}^+(\hat{X}(q,t),t)$ and since $\bar{x}^{\pm}$ may be singular at the crack-front it does not follow that $\hat{x}$ is differentiable with respect to t .

Thus we need

K8: The crack-front in the spatial description F_t , $t \in T$ is a moving curve in E with normal velocity $\bar{c}$.

We now want to establish a relation between the velocities $\bar{c}$, $\bar{c}$ and $\dot{x}^{\pm}$. For this we need some additional assumptions.

Proposition 2.3 Let $\chi(\cdot,t)$, $t \in T$ be a family of fracture-deformations satisfying the items K1 - K8 stated above and in addition

K9: There is a fixed smooth surface S_e in E and mappings

$$S_e \times T \ni (X,t) \leadsto \bar{X}_e^{\pm}(X,t) \in E$$

of class C^2 satisfying: $\tilde{S}_t \subset S_e$ for every $t \in T$, $\bar{X}^+(X,t) = \bar{X}_e^+(X,t)$ and $\bar{X}^-(X,t) = \bar{X}_e^-(X,t)$, $X \in \tilde{S}_t$ (i.e. $\bar{X}^{\pm}(\cdot,t)$ are restrictions of $\bar{X}_e^{\pm}(\cdot,t)$ to $\tilde{S}_t$).

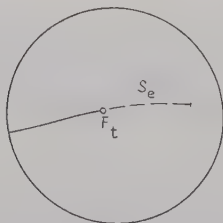


Figure 2.9

Then the velocity fields $\dot{x}^\pm(\cdot, t)$ may be extended to the crack-front by their finite limits as $\tilde{F}_t$ is approached from $\tilde{S}_t \setminus \tilde{F}_t$ and

$$(\tilde{c}_t^\pm(x) - \dot{x}^\pm(x, t)) \cdot \tilde{v}_t^\pm(x) = \tilde{\zeta}_t(X)(L_t(X))^{-1} A^\pm(X, t) \quad (2.29)$$

$x = X^\pm(X, t)$, $X \in \tilde{F}_t$. Here $\tilde{v}_t^\pm = \tilde{v}_t^\pm(x) \in T_x^\pm$ is the outward pointing (relative to $\tilde{S}_t^\pm$) unit normal on $\tilde{F}_t$ (see figure below) and $L = L_t(X)$, $X \in \tilde{F}_t$ is the length ratio associated with the restriction of $X^\pm(\cdot, t)$ to $\tilde{F}_t$. Furthermore let $\tilde{e} = \tilde{e}_t(x)$, $x \in \tilde{F}_t$ denote a unit tangent vector on $\tilde{F}_t$ and put

$$j = j_t(x) \equiv |\tilde{m}_t^+(x) \times \tilde{m}_t^-(x) \cdot \tilde{e}_t(x)|, \quad x \in \tilde{F}_t$$

If

a) $j \neq 0$ then we have (arguments are suppressed, $X \in \tilde{F}_t$)

$$\begin{aligned} \tilde{c} = & -\frac{1}{j} (\tilde{m}^- \cdot \dot{x}^-) \tilde{v}^+ - \frac{1}{j} (\tilde{m}^+ \cdot \dot{x}^+) \tilde{v}^- = -\frac{1}{j} (\tilde{v}^- \cdot \dot{x}^- + \tilde{\zeta} L^{-1} A^-) \tilde{m}^+ - \\ & \frac{1}{j} (\tilde{v}^+ \cdot \dot{x}^+ + \tilde{\zeta} L^{-1} A^+) \tilde{m}^- \end{aligned} \quad (2.30)$$

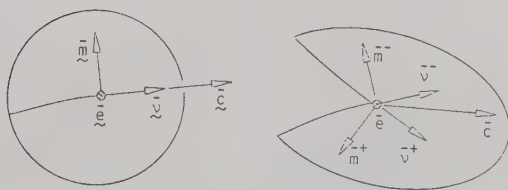


Figure 2.10

b) $j = 0$ and we have a sharp crack-tip ($\tilde{v}^+ = \tilde{v}^- = \tilde{v}$, $\tilde{m}^+ = -\tilde{m}^- = \tilde{m}$) then

$$\tilde{c} = (\tilde{v} \cdot \dot{x}^\pm + \tilde{\zeta} L^{-1} A^\pm) \tilde{v} + (\tilde{m} \cdot \dot{x}^\pm) \tilde{m} \quad (2.31)$$

c) $j = 0$ and we have a blunt crack-tip ($\bar{v}^+ = -\bar{v}^- = \bar{v}$, $\bar{m}^+ = \bar{m}^- = \bar{m}$) then

$$\bar{c} = (\bar{v} \cdot \dot{\bar{x}}^\pm \pm \underline{c} L^{-1} A^\pm) \bar{v} + (\bar{m} \cdot \dot{\bar{x}}^\pm) \bar{m} \quad (2.32)$$

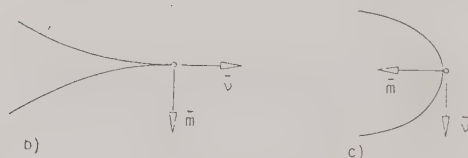


Figure 2.11

Proof: Let $\hat{X}$ be a parametrization of the crack-front in the referential description, i.e.

$$F_t = \{X \in E \mid X = \hat{X}(q, t), q \in [a, b]\}, \quad t \in T$$

then, in the spatial description, we have

$$F_t = \{x \in E \mid x = \hat{x}(q, t), q \in [a, b]\}, \quad t \in T$$

where

$$\hat{x}(q, t) = \bar{x}^\pm(\hat{X}(q, t), t) = \bar{\chi}_e^\pm(\hat{X}(q, t), t)$$

Let $\bar{e} = \bar{e}_t(x)$, $x \in F_t$ denote a unit tangent vector on F_t . Then we have (cf. Appendix A.5)

$$\bar{c} = \frac{\partial}{\partial t} \hat{x} - (\bar{e} \cdot \frac{\partial}{\partial t} \hat{x}) \bar{e} \quad (2.33)$$

and by the chain rule we get (since $\bar{\chi}_e^\pm$ are smooth on $S_e \times T$)

$$\frac{\partial}{\partial t} \hat{x} = \frac{\partial}{\partial t} \bar{\chi}_e^\pm(\hat{X}(q, t), t) = F_e^\pm \frac{\partial}{\partial t} \hat{X} + \dot{\bar{\chi}}_e^\pm = F_e^\pm \frac{\partial}{\partial t} \hat{X} + \dot{\bar{x}}^\pm \quad (2.34)$$

where $\underline{F}^{\pm}$ and $\dot{\underline{x}}^{\pm}$ are evaluated at $X = \hat{X}(q, t)$. But

$$\frac{\partial}{\partial t} \hat{X} = \underline{\bar{c}} + (\underline{\bar{e}} \cdot \frac{\partial}{\partial t} \hat{X}) \underline{\bar{e}} \quad (2.35)$$

where $\underline{\bar{e}} = \underline{\bar{e}}_t(X)$ is the unit tangent vector on $\underline{F}_t$ satisfying

$$\underline{F}^{\pm} \underline{\bar{e}} = L \underline{\bar{e}} \quad (2.36)$$

From (2.33) - (2.36) we get

$$\underline{\bar{c}} = \underline{\bar{c}} \underline{F}^{\pm} \underline{\bar{v}} + \dot{\underline{x}}^{\pm} + (L \underline{\bar{e}} \cdot \frac{\partial}{\partial t} \hat{X} - \underline{\bar{e}} \cdot \frac{\partial}{\partial t} \hat{X}) \underline{\bar{e}} \quad (2.37)$$

and then

$$(\underline{\bar{c}} - \dot{\underline{x}}^{\pm}) \cdot \underline{\bar{v}}^{\pm} = \underline{\bar{c}} \underline{\bar{v}}^{\pm} \cdot \underline{F}^{\pm} \underline{\bar{v}} \quad (2.38)$$

But we have the geometric relation (see section 3.2, Lemma 3.3)

$$\underline{\bar{v}}^{\pm} = L^{-1} A^{\pm} (\underline{F}^{\pm})^{-T} \underline{\bar{v}} \quad (2.39)$$

and formula (2.29) now follows from (2.38) and (2.39). For the proof of (2.30) we note that

$$\underline{\bar{m}}^{\pm} \cdot \underline{\bar{c}} = \underline{\bar{m}}^{\pm} \cdot \dot{\underline{x}}^{\pm}$$

$$\underline{\bar{v}}^{\pm} \cdot \underline{\bar{c}} = \underline{\bar{v}}^{\pm} \cdot \dot{\underline{x}}^{\pm} + \underline{\bar{c}} L^{-1} A^{\pm}$$

and that if $j = |\underline{\bar{m}}^+ \times \underline{\bar{m}}^- \cdot \underline{\bar{e}}| \neq 0$ then the basis

$$(-\frac{1}{j} \underline{\bar{v}}^- - \frac{1}{j} \underline{\bar{v}}^+ \cdot \underline{\bar{e}}) \text{ is reciprocal to } (\underline{\bar{m}}^+ \quad \underline{\bar{m}}^- \quad \underline{\bar{e}})^{-1}$$

In b) and c) we have expressed $\underline{\bar{c}}$ in the orthonormal basis $(\underline{\bar{v}} \quad \underline{\bar{m}} \quad \underline{\bar{e}})$. $\square$

Corollary 2.1 Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K9 and

- a) assume that we have a sharp crack-tip at $X \in \underline{F}_t$ and that $A^+(X, t) = A^-(X, t)$. Then the jump in material velocity at the crack-front $\dot{\underline{x}}^+(X, t) - \dot{\underline{x}}^-(X, t)$ is parallel to $\underline{\bar{e}}_t(x)$, $x = \hat{X}^{\pm}(X, t)$ the unit

tangent vector on F_t .

- b) assume that we have a blunt crack-tip at $X \in F_t$ and that the crack is advancing, i.e. $\zeta_t(X) > 0$, then

$$\bar{m}_t(X) \cdot (\dot{x}^+(X, t) - \dot{x}^-(X, t)) = 0$$

$$\bar{v}_t(X) \cdot (\dot{x}^+(X, t) - \dot{x}^-(X, t)) < 0$$

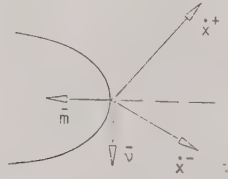


Figure 2.12

Proof: a) From formula (2.31) it follows that

$$\bar{m} \cdot (\dot{x}^+ - \dot{x}^-) = 0$$

$$\bar{v} \cdot (\dot{x}^+ - \dot{x}^-) = -\zeta L^{-1}(A^+ - A^-)$$

and since $A^+ = A^-$ we conclude that $\dot{x}^+ - \dot{x}^-$ is parallel to $\bar{e}$.

b) From (2.32) it follows that

$$\bar{m} \cdot (\dot{x}^+ - \dot{x}^-) = 0$$

$$\bar{v} \cdot (\dot{x}^+ - \dot{x}^-) = -\zeta L^{-1}(A^+ + A^-)$$

and since $\zeta > 0$, $L > 0$ and $A^\pm > 0$ we conclude that $\bar{v} \cdot (\dot{x}^+ - \dot{x}^-) < 0$.

□

Remark: Note that we may always write (for $j \geq 0$)

$$\bar{c} = (\bar{v}^\pm \cdot \dot{x}^\pm + \zeta L^{-1} A^\pm) \bar{v}^\pm + (\bar{m}^\pm \cdot \dot{x}^\pm) \bar{m}^\pm \quad (2.40)$$

There are infinitely many possible values for $\bar{\zeta}$ corresponding to different choices of reference configuration and thus the physical significance of the crack-front velocity in the referential description is rather weak. The scalar fields

$$c^\pm = c_t^\pm(x) \equiv (\bar{c}_t(x) - \dot{x}^\pm(x,t)) \cdot \bar{v}_t^\pm(x), \quad x \in F_t \quad (2.41)$$

however, do not depend on the reference configuration. They will be called intrinsic crack-front speeds and from (2.29) in Proposition 2.3 we have the relation

$$c_t^\pm(x) = \zeta_t(X)(L_t(X))^{-1} A^\pm(X,t), \quad x = X^\pm(X,t), \quad X \in \tilde{F}_t$$

This formula was demonstrated under the assumption that X satisfies K9. If this condition fails, in the sense that $\dot{x}^\pm$ are unbounded at the crack-front, then formula (2.29) may loose its meaning. Note, however, that the intrinsic crack-front speeds are well-defined provided that the tangential velocity fields $\dot{x}_{||}^\pm$ may be extended to the crack-front. In order to motivate the extension of formula (2.29) to fracture-deformations with $\dot{x}^\pm \cdot \bar{m}^\pm$ singular at the crack-front we consider an advancing crack, i.e. $\zeta_t(x) > 0$ for $X \in \tilde{F}_t$, $t \in T$. For a fixed $\delta > 0$ the moving curves

$$F_{t,\delta}^\pm \equiv X^\pm(F_{t-\delta}, t), \quad t \in T$$

have normal velocities $\bar{c}_{t,\delta}^\pm$ satisfying

$$(\bar{c}_{t,\delta}^\pm(z^\pm) - \dot{x}^\pm(z^\pm, t)) \cdot \bar{v}_{t,\delta}^\pm(z^\pm) = \zeta_{t-\delta}(Z)(L_{t,\delta}(Z))^{-1} A^\pm(Z, t) \quad (2.42)$$

$z^\pm = X^\pm(Z, t)$, $Z \in \tilde{F}_{t-\delta}$. Here $\bar{v}_{t,\delta}^\pm$ is the outward pointing (relative to $S_{t,\delta}^\pm \equiv X^\pm(S_{t-\delta}, t)$) unit normal on $\tilde{F}_{t-\delta}$ and $L_{t,\delta}$ is the length ratio associated with the restriction of $X^\pm(\cdot, t)$ to $\tilde{F}_{t-\delta}$. The proof of (2.42) is analogous to the proof of (2.39).

Proposition 2.3' Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K8 and in addition

K9': The tangential velocity fields $\dot{x}_{||}^\pm$ may be extended to the crack-front by their finite limits as $\tilde{F}_{t-\delta}$ is approached from $S_{t-\delta} \setminus \tilde{F}_{t-\delta}$.

The crack is advancing and for $X \in \tilde{F}_t$ we have

$$\lim_{\substack{\delta \rightarrow 0 \\ \tilde{F}_{t-\delta} \ni Z \rightarrow X}} \bar{c}_{t,\delta}^{\pm}(\chi^{\pm}(Z,t)) = \bar{c}_t^{\pm}(\chi^{\pm}(X,t)) \quad (2.43)$$

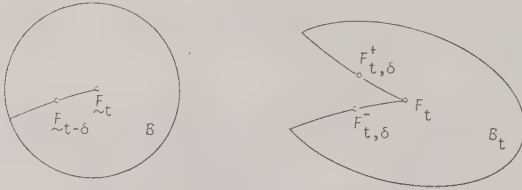


Figure 2.13

Then formula (2.29), with $\dot{x}^{\pm}$ substituted for $\dot{x}_{||}^{\pm}$, is valid.

Proof: Formula (2.29) is obtained by taking limits in (2.42). Note that $(\bar{c}_{t,\delta}^{\pm} - \dot{x}^{\pm}) \cdot \bar{v}_{t,\delta}^{\pm} = (\bar{c}_{t,\delta}^{\pm} - \dot{x}_{||}^{\pm}) \cdot \bar{v}_{t,\delta}^{\pm}$. $\square$

Remark: Similar arguments will apply to a retreating crack, i.e.

$\underline{c}_t(X) < 0$ for $X \in \tilde{F}_t$, and a resting crack, i.e. $\underline{c}_t(X) = 0$ for $X \in \tilde{F}_t$, or more general to a crack that is locally advancing, retreating or resting.

2.2 Fracture field and crack-surface field

Consider a family of fields

$$\Phi(\cdot, t) : \bar{B} \setminus \tilde{S}_t \rightarrow \mathbb{R}, V, \text{End}(V), \dots \quad t \in T \quad (2.44)$$

defined on a fracturing body. In a concrete thermo-mechanical application one may think of Φ as representing temperature, material velocity, stress tensor etc. We write

$$\Phi \in L^p(B_t), \quad 0 < p < \infty$$

if $\Phi(\cdot, t)$ is measurable on $B_t \equiv \chi(\bar{B} \setminus \underline{S}_t, t)$ (Lebesgue volume measure in E) and

$$\int_{B_t} |\Phi(x, t)|^p dv(x) < \infty$$

for every $t \in T$.

Remark 1: By following a standard abuse of notation in continuum mechanics the single symbol Φ is used to denote our field in the referential as well as in the spatial description. We write

$$\Phi \in L^p(B), \quad 0 < p < \infty$$

if $\Phi(\cdot, t)$ is measurable on B (or $\bar{B} \setminus \underline{S}_t$) and

$$\int_B |\Phi(X, t)|^p dv(X) < \infty$$

for every $t \in T$.

Remark 2: The irregularities of Φ will, in general, be assumed to be localized to the crack-front (unboundedness) and to a finite number of singular surfaces (jump-discontinuities).

Let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations. On account of the properties of χ , as established in the previous section, we have

Proposition 2.4 $\Phi \in L^1(B_t)$ if and only if $\Phi_p \equiv J\Phi \in L^1(B)$ and then we have

$$\int_B \Phi_p(X, t) dv(X) = \int_{B_t} \Phi(x, t) dv(x) \quad (2.45)$$

Here J is the volume-ratio defined in (2.12) and Φ_p is called the Piola-field corresponding to Φ .

Proof: The change of variable formula. $\square$ ¹

Corollary 2.2 $J \in L^1(B)$.

Proof: Take $\Phi \equiv 1$ in Proposition 2.4 and note that in K1 we have assumed that B_t is a bounded set in E . $\square$

¹ Schwartz [8]

The family of fields in (2.44) is called a C^k -fracture-field if

F1: for every open subset B^- of B and every open subset T^- of T satisfying $B^- \cap \underset{t}{S}_t = \emptyset$ for all $t \in T^-$, the mapping

$$B^- \times T^- \ni (X, t) \leadsto \Phi(X, t)$$

is C^k .

F2: the limits in the sense of (2.1) exist for Φ and all its derivatives.

and a C^k -fracture-density if in addition

$$\text{F3: } \Phi \in L^1(S_t)$$

Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1-K8 and let Φ be a C^1 -fracture-field. The referential derivatives of the corresponding Piola-field Φ_p are given by

Lemma 2.5

$$\begin{aligned}\dot{\Phi}_p &= J(\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \\ \operatorname{Grad} \Phi_p &= J(\operatorname{Grad} \Phi + \Phi \otimes \operatorname{div} F^T)\end{aligned}\tag{2.46}$$

Proof: Differentiating the relation $\Phi_p = J\Phi$ gives

$$\begin{aligned}\dot{\Phi}_p &= \dot{J}\Phi + J\dot{\Phi} = J(\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \\ \operatorname{Grad} \Phi_p &= J \operatorname{Grad} \Phi + \Phi \otimes \operatorname{Grad} J = J(\operatorname{Grad} \Phi + \Phi \otimes \operatorname{div} F^T)\end{aligned}$$

where we have used (2.18)_{1,2} in Lemma 2.3. $\square$

Remark: The identities in Lemma 2.5 may also be written

$$\begin{aligned}\dot{\Phi}_p &= J\left(\frac{\partial}{\partial t}\Phi + \operatorname{div}(\Phi \otimes \dot{X})\right) \\ \operatorname{Grad} \Phi_p &= J((\operatorname{grad} \Phi)F + \Phi \otimes \operatorname{div} F^T)^{-1}\end{aligned}\tag{2.47}$$

The differential operators: $\cdot$, $\frac{\partial}{\partial t}$, Grad and grad are introduced in chapter 1.

$\cdot$: Φ vector-valued.

Let Φ be a fracture-density and consider the function

$$T \ni t \mapsto \int_{B_t} \Phi(x, t) dv(x) \quad (2.48)$$

In a thermo-mechanical application $\Phi(\cdot, t)$ may, for instance, represent the energy per unit volume in the present configuration B_t . Since Φ may be singular at the crack-front the differentiability properties of (2.48) are unclear. We have the following elementary result based on the fact that $\bigcup_{t \in T} \tilde{S}_t$ is a set of volume-measure zero.

Proposition 2.5 Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K8 and let Φ be a C^1 -fracture-density.

If there is a (time-independent, real-valued and non-negative) function $f \in L^1(B)$ satisfying

$$|\dot{\Phi}_p(X, t)| \leq f(X), \quad X \in B \setminus \tilde{S}_t, \quad t \in T \quad (2.49)$$

then the function (2.48) is C^1 and

$$\frac{d}{dt} \int_{B_t} \Phi dv(x) = \int_{B_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X}) dv(x) \quad (2.50)$$

Proof: The function (2.48) may be written

$$T \ni t \mapsto \int_B \Phi_p(X, t) dv(X) \quad (2.51)$$

and since $\tilde{S}^* = \bigcup_{t \in T} \tilde{S}_t$ is a set of volume-measure zero,

$$|\dot{\Phi}_p(X, t)| \leq f(X), \quad X \in B \setminus \tilde{S}^*, \quad t \in T$$

and $\dot{\Phi}_p(X, \cdot)$ is continuous on T for every $X \in B \setminus \tilde{S}^*$ we may conclude (by using a standard theorem in integral calculus¹) that (2.51) is a C^1 -function and

$$\frac{d}{dt} \int_B \Phi_p dv(X) = \int_B \dot{\Phi}_p dv(X) \quad (2.52)$$

Formula (2.50) now follows from (2.52) and (2.46)₁ in Lemma 2.5. $\square$

Remark: Condition (2.49) is fulfilled if, for instance

¹ see Schwartz [8], chap.1, Theorem 46.

$$J(X, t) \leq g(X)$$

$$|(\dot{\Phi} + \Phi \operatorname{div} \dot{X})(X, t)| \leq h(X) \quad X \in B \setminus \tilde{S}_t, \quad t \in T$$

where $g \in L^p(B)$ and $h \in L^q(B)$ ($1/p + 1/q = 1$, $1 \leq p \leq \infty$)

Following Gurtin & Yatomi [9] we write

$$\Phi \in L^p(B_t) \text{ uniformly in time}$$

if $\Phi \in L^p(B_t)$ and if, given any one-parameter family $B_{t,\delta}$, $\delta > 0$ consisting of open subsets of B_t and with $\operatorname{volume}(B_t \setminus B_{t,\delta}) \rightarrow 0$ as $\delta \rightarrow 0$, we have

$$\int_{B_{t,\delta}} |\Phi(x, t)|^p dv(x) \rightarrow \int_{B_t} |\Phi(x, t)|^p dv(x)$$

uniformly in $t \in T$ as $\delta \rightarrow 0$.

This definition is adapted to the fact that

$$\tilde{S}^* \equiv \bigcup_{t \in T} \tilde{S}_t \quad (2.53)$$

is a set of volume-measure zero. We assume that there is a family R_δ , $\delta > 0$ of open sets in E satisfying

$$\tilde{S}^* \subset R_\delta, \quad \delta > 0 \text{ and } \operatorname{volume}(R_\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0 \quad (2.54)$$

As an alternative to Proposition 2.5 we have

Proposition 2.6 Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K8 and let Φ be a C^1 -fracture-density. Assume that

$$\dot{\Phi} + \Phi \operatorname{div} \dot{X} \in L^1(B_t) \text{ uniformly in time}$$

Then (2.48) is a C^1 -function and formula (2.50) is valid.

Proof: Take R_δ as in (2.54). Put $B_\delta \equiv B \setminus R_\delta$. Then $B_{t,\delta} \equiv X(B_\delta, t)$, $\delta > 0$ is a family of open sets in B_t with

$$\text{volume}(B_t \setminus B_{t,\delta}) = \int_{B \setminus B_\delta} J(X,t) \, dv(X) \rightarrow 0 \text{ as } \delta \rightarrow 0$$

(since $J \in L^1(B)$ and $\text{volume}(B \setminus B_\delta) \rightarrow 0$ as $\delta \rightarrow 0$). We may then conclude that

$$\frac{d}{dt} \int_{B_{t,\delta}} \Phi \, dv(x) = \frac{d}{dt} \int_{B_\delta} \Phi_P \, dv(X) = \int_{B_\delta} \dot{\Phi}_P \, dv(X) = \int_{B_{t,\delta}} (\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \, dv(x) \rightarrow$$

$$\int_{B_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \, dv(x) \text{ uniformly in time as } \delta \rightarrow 0$$

But since

$$\int_{B_{t,\delta}} \Phi \, dv(x) \rightarrow \int_{B_t} \Phi \, dv(x) \text{ (pointwise in } t \in T \text{) as } \delta \rightarrow 0$$

and the functions

$$T \ni t \mapsto \int_{B_{t,\delta}} (\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \, dv(x), \quad \delta > 0$$

are continuous, the Proposition follows from standard theorems on uniform convergence.¹ $\square$

Remark: The function (2.48) will enter the global balance law to be postulated in chapter 3 and we will need the assumption that it is a differentiable function. However, since the requirements (on X and Φ) stated in Propositions 2.5 and 2.6 are not generally met with, we may have

$$\frac{d}{dt} \int_{B_t} \Phi \, dv(x) \neq \int_{B_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X}) \, dv(x)$$

We will give this problem further attention in section 2.3.

Consider the two families of fields

$$\psi^\pm(\cdot, t) : S_t \setminus F_t \rightarrow \mathbb{R}, \, V, \operatorname{End}(V), T^\pm, \dots \quad t \in T \quad (2.55)$$

defined on the crack-surfaces. We write

$$\psi^\pm \in L^p(S_t^\pm)$$

¹ cf. Schwartz [8], chap.1, Theorems 39 and 43.

if $\Psi^\pm(\cdot, t)$ is measurable on $S_t^\pm$ (Lebesgue area-measure in E) and

$$\int_{S_t^\pm} |\Psi^\pm(x, t)|^p da(x) < \infty$$

for every $t \in T$.

Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations (in K2 we have assumed that $X^\pm(\cdot, t) : \tilde{S}_t^\pm \rightarrow S_t^\pm$ are diffeomorphisms). Then, analogous to Proposition 2.4, we have

Proposition 2.7 $\Psi^\pm \in L^1(S_t^\pm)$ if and only if $\Psi_p^\pm \equiv A^\pm \Psi^\pm \in L^1(\tilde{S}_t^\pm)$ and then we have

$$\int_{\tilde{S}_t^\pm} \Psi_p^\pm(X, t) da(X) = \int_{S_t^\pm} \Psi^\pm(x, t) da(x)$$

Here $A^\pm$ is the area ratio defined in (2.13) and $\Psi_p^\pm$ is called the Piola-field corresponding to $\Psi^\pm$.

The families of fields in (2.55) are called C^k -crack-surface-fields if

F4: $\Psi^\pm(\cdot, t)$ may be continuously extended to the crack-front for every $t \in T$.

F5: for every open subset T^- of T and every S^- , open in $\tilde{S}_t^- \setminus F_t$ for all $t \in T^-$, the mappings

$$S^- \times T^- \ni (X, t) \mapsto \Psi^\pm(X, t)$$

are C^k .

Remark: Note that if $\Psi^\pm$ is a crack-surface-field then $\Psi^\pm \in L^1(S_t^\pm)$.

Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1-K8 and let $\Psi^\pm$ be C^1 -crack-surface-fields. Then, analogous to Lemma 2.5, we have

Lemma 2.6

$$\begin{aligned} \dot{\Psi}_p^\pm &= A^\pm (\dot{\Psi}^\pm + \Psi^\pm \operatorname{div}_\delta \dot{X}^\pm) \\ \operatorname{Grad}_\delta \Psi_p^\pm &= A^\pm (\operatorname{Grad}_\delta \Psi^\pm + \Psi^\pm \otimes (\operatorname{div}_\delta (F_t^\pm)^T - 2M\tilde{m})) \end{aligned} \quad (2.56)$$

Proof: A straightforward calculation using (2.18)_{3,4}. $\square$

Let $\Psi^\pm$ be crack-surface-fields and consider the functions

$$T \ni t \leadsto \int_{S_t^\pm} \Psi^\pm(x, t) da(x) \quad (2.57)$$

In a thermo-mechanical application $\Psi^\pm(\cdot, t)$ may, for instance, represent the (surface-) energy per unit area in the present configuration.

Proposition 2.8 Let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K9 and let $\Psi^\pm$ be C^1 -crack-surface-fields satisfying

F6: With S_e as in K9 there are C^1 -functions

$$S_e \times T \ni (X, t) \leadsto \Psi_e^\pm(X, t)$$

satisfying: $\Psi^\pm(X, t) = \Psi_e^\pm(X, t)$ for $X \in \tilde{S}_t$, $t \in T$.

then the functions (2.57) are C^1 and

$$\frac{d}{dt} \int_{S_t^\pm} \Psi^\pm da(x) = \int_{S_t^\pm} (\dot{\Psi}^\pm + \Psi^\pm \operatorname{div}_\Delta \dot{X}^\pm) da(x) + \int_{F_t} \Psi^\pm c^\pm ds(x) \quad (2.58)$$

(Here ds denotes the arc-length-measure in E and $c^\pm$ are the intrinsic crack-front speeds defined in (2.41)).

Proof: We have

$$\frac{d}{dt} \int_{S_t^\pm} \Psi^\pm da(x) = \frac{d}{dt} \int_{\tilde{S}_t} \Psi_p^\pm da(X) = \frac{d}{dt} \int_{\tilde{S}_t} \Psi_{ep}^\pm da(X) = \int_{\tilde{S}_t} \dot{\Psi}_{ep}^\pm da(X) + \int_{\tilde{F}_t} \Psi_{ep}^\pm \tilde{c} da(X) =$$

$$\int_{\tilde{S}_t} \dot{\Psi}_p^\pm dv(X) + \int_{\tilde{F}_t} \Psi_p^\pm \tilde{c} ds(X) = \int_{S_t^\pm} (\dot{\Psi}^\pm + \Psi^\pm \operatorname{div}_\Delta \dot{X}^\pm) da(x) + \int_{F_t} \Psi^\pm c^\pm ds(x)$$

where we, in the third step, have used "the surface transport theorem" (presented in Appendix A.9) and, in the final step, the relations (2.29) and (2.56)₁. $\square$

We may drop the requirement that $\dot{\Psi}^\pm$ and $\operatorname{div}_\Delta \dot{X}^\pm$ are continuously extendable to the crack-front if we assume some uniformity in time.

We write

$$\psi^\pm \in L^P(S_t^\pm) \text{ uniformly in time}$$

if $\psi^\pm \in L^P(S_t^\pm)$ and if, given any one-parameter family $S_{t,\delta}^\pm$, $\delta > 0$ consisting of open subsets of $S_t^\pm$ and with $\text{area}(S_t^\pm \setminus S_{t,\delta}^\pm) \rightarrow 0$ as $\delta \rightarrow 0$, we have

$$\int_{S_{t,\delta}^\pm} |\psi^\pm(x,t)|^P da(x) \rightarrow \int_{S_t^\pm} |\psi^\pm(x,t)|^P da(x)$$

uniformly in $t \in T$ as $\delta \rightarrow 0$.

Proposition 2.8' Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying K1 - K8, K9'. We assume that the crack is advancing and

$$\text{K9'': } \text{div}_\delta \dot{x}^\pm \in L^1(S_t^\pm) \text{ uniformly in time}$$

Let $\psi^\pm$ be C^1 -crack-surface-fields satisfying

$$\text{F6': } \dot{\psi}^\pm \in L^1(S_t^\pm) \text{ uniformly in time}$$

Then (2.57) is a C^1 -function and formula (2.58) is valid.

Proof: Let T^- be a closed subinterval of T and consider, for fixed $\delta > 0$, the function

$$T^- \ni t \mapsto \int_{S_{t,\delta}^\pm} \psi^\pm(x,t) da(x)$$

where

$$S_{t,\delta}^\pm \equiv X^\pm(\tilde{S}_{t-\delta}, t), \quad \delta > 0$$

An application of the surface transport theorem (as in the proof of Proposition 2.8) gives

$$\frac{d}{dt} \int_{S_{t,\delta}^\pm} \psi^\pm(x,t) da(x) = \int_{\tilde{S}_{t-\delta}} \dot{\psi}_p^\pm(X,t) da(X) + \int_{\tilde{F}_{t-\delta}} \psi_p^\pm(X,t) \mathcal{C}_{t-\delta}(X) da(X)$$

and we have the estimate

$$\begin{aligned}
& \left| \frac{d}{dt} \int_{S_{t,\delta}^{\pm}} \psi^{\pm} da(x) - \int_{S_t^{\pm}} (\dot{\psi}^{\pm} + \psi^{\pm} \operatorname{div}_{\Delta} \dot{x}^{\pm}) da(x) - \int_{F_t} \psi^{\pm} c^{\pm} ds(x) \right| \leq \\
& \int_{S_t^{\pm} \setminus S_{t,\delta}^{\pm}} |\dot{\psi}^{\pm}| da(x) + C \left(\int_{S_t^{\pm} \setminus S_{t,\delta}^{\pm}} |\operatorname{div}_{\Delta} \dot{x}^{\pm}| da(x) \right) + \Delta^{\pm}(t, \delta)
\end{aligned} \quad (2.59)$$

where C is a constant satisfying

$$\sup_{t \in T^-} \sup_{x \in S_t^{\pm}} |\psi^{\pm}(x, t)| \leq C$$

and (for some "small" $\alpha > 0$)

$$T^- \times [0, \alpha] \ni (t, \delta) \leadsto \Delta^{\pm}(t, \delta) \equiv \left| \int_{F_t} \psi_p^{\pm}(X, t) \tilde{c}_t(X) ds(X) - \right.$$

$$\left. \int_{F_{t-\delta}} \psi_p^{\pm}(X, t) \tilde{c}_{t-\delta}(X) ds(X) \right|$$

are continuous functions satisfying (note that $\Delta^{\pm}(t, 0) = 0$, $t \in T^-$)

$$\lim_{\delta \rightarrow 0} \sup_{t \in T^-} \Delta^{\pm}(t, \delta) = 0 \quad (2.60)$$

Now, since

$$\lim_{\delta \rightarrow 0} \operatorname{area}(S_t^{\pm} \setminus S_{t,\delta}^{\pm}) = 0$$

we may conclude, from (2.59), K9", F6' and (2.60), that

$$\frac{d}{dt} \int_{S_{t,\delta}^{\pm}} \psi^{\pm} da(x) \rightarrow \int_{S_t^{\pm}} (\dot{\psi}^{\pm} + \psi^{\pm} \operatorname{div}_{\Delta} \dot{x}^{\pm}) da(x) + \int_{F_t} \psi^{\pm} c^{\pm} ds(x)$$

uniformly in $t \in T^-$ as $\delta \rightarrow 0$. Of course we have

$$\int_{S_{t,\delta}^{\pm}} \psi^{\pm} da(x) \rightarrow \int_{S_t^{\pm}} \psi^{\pm} da(x)$$

pointwise in $t \in T^-$ as $\delta \rightarrow 0$ and since the functions

$$T^- \ni t \leadsto \int_{S_{t-\delta}^{\pm}} \psi_p^{\pm} da(X) + \int_{F_{t-\delta}} \psi_p^{\pm} \tilde{c}_{t-\delta} ds(X)$$

are continuous the Proposition follows from standard theorems on uniform

convergence. $\square$

Remark 1: With $\Psi^\pm \equiv 1$ in formula (2.58) we obtain the growth-rate in crack-surface area:

$$\frac{d}{dt} \int_{S_t^\pm} da(x) = \int_{S_t^\pm} \operatorname{div}_\Delta \dot{x}^\pm da(x) + \int_{F_t} c^\pm da(x)$$

Remark 2: The conclusion in Proposition 2.8' is still valid if $K9''$ is substituted for

K9''': The fields $\operatorname{div}_\Delta \dot{x}_{||}^\pm(\cdot, t)$, $t \in T$ may be continuously extended to the crack-front and the normal velocity field $\dot{x}^\pm \cdot \bar{m}^\pm \in L^1(S_t^\pm)$ uniformly in time.

This follows from the simple estimate ($t \in T^-$ and $\delta > 0$)

$$\int_{S_t^\pm \setminus S_{t,\delta}^\pm} |\operatorname{div}_\Delta \dot{x}^\pm| da(x) \leq C \left(\int_{S_t^\pm \setminus S_{t,\delta}^\pm} |\dot{x}^\pm \cdot \bar{m}^\pm| da(x) + \operatorname{area}(S_t^\pm \setminus S_{t,\delta}^\pm) \right)$$

where C is a constant satisfying

$$C \geq \max \left(\sup_{t \in T^-} \sup_{x \in S_t^\pm} |M^\pm|, \sup_{t \in T^-} \sup_{x \in S_t^\pm} |\operatorname{div}_\Delta \dot{x}_{||}^\pm| \right)$$

and the fact that

$$\lim_{\delta \rightarrow 0} \sup_{t \in T^-} \operatorname{area}(S_t^\pm \setminus S_{t,\delta}^\pm) = 0$$

2.3 The tubular neighbourhood of the crack-front.

In fracture mechanics the crack-front and its immediate vicinity attracts special attention. In this section we introduce a useful set of coordinates for the region close to the crack-front. We also discuss a convective derivative associated with the crack-front motion.

Consider the crack-front $\underline{F}_t$, $t \in T$ in the referential description. We assume that $\underline{F}_t$ has the global parametrization:

$$\underline{F}_t = \{X \in E \mid X = \hat{X}(s, t), s \in [s_1(t), s_2(t)]\}, \quad t \in T$$

where s is the arc-length coordinate on $\tilde{F}_t$ (corresponding to a given orientation). Let $\tilde{e} = \tilde{e}_t(s)$, $\tilde{p} = \tilde{p}_t(s)$ and $\tilde{b} = \tilde{b}_t(s)$ denote respectively the unit tangent, the principal normal and the binormal associated with the space curve $\tilde{F}_t$. Let $\tilde{\kappa} = \tilde{\kappa}_t(s)$ and $\tilde{\tau} = \tilde{\tau}_t(s)$ denote the curvature and torsion of $\tilde{F}_t$ (see Appendix A.5).

We fix a time $t \in T$ and ignore, for the present, the time-parameter in our formulas. Put, for $r_0 > 0$,

$$D(r_0) \equiv \{(q_1, q_2) \in \mathbb{R}^2 \mid 0 < q_1^2 + q_2^2 < r_0^2\}$$

A tubular neighbourhood $\tilde{T}N(r_0)$ to $\tilde{F}$ with radius r_0 is defined by

$$\begin{aligned} \tilde{T}N(r_0) \equiv \{X \in E \mid X = \tilde{X}(s, q_1, q_2) \equiv \hat{X}(s) + q_1 \tilde{p}(s) + q_2 \tilde{b}(s), s \in \\ [s_1, s_2], (q_1, q_2) \in D(r_0)\} \end{aligned} \quad (2.61)$$

Note that the tubular neighbourhood does not contain the crack-front. Put

$$\kappa_0 \equiv \sup_{s \in [s_1, s_2]} \tilde{\kappa}(s), \quad \tau_0 \equiv \sup_{s \in [s_1, s_2]} \tilde{\tau}(s)$$

Lemma 2.7 If

$$0 < r_0 < \kappa_0^{-1} \quad (2.62)$$

then the mapping $\tilde{X}: \mathbb{R}^3 \supset [s_1, s_2] \times D(r_0) \rightarrow \tilde{T}N(r_0)$, defined in (2.61), is a diffeomorphism, i.e. we may use (s, q_1, q_2) as global coordinates in the tubular neighbourhood.

Proof: It is obvious that $\tilde{X}$ is a smooth mapping. Let Q denote the matrix of $\nabla \tilde{X}$ with respect to the natural basis in $\mathbb{R}^3$ and the basis $(\tilde{e}, \tilde{p}, \tilde{b})$ in V . By a straightforward calculation, using the Frenet formula (A.7), we get

$$\det Q(s, q_1, q_2) = 1 - \tilde{\kappa}(s) q_1 > 1 - \kappa_0 r_0 > 0$$

for all $(s, q_1, q_2) \in [s_1, s_2] \times D(r_0)$ where, in the second inequality, we have used (2.62).

We also have

$$|(\nabla \bar{X}(s, q_1, q_2))^{-1}| = \frac{\sqrt{1 + \tau(s)^2(q_1^2 + q_2^2) + 2(1 - \kappa(s)q_1)^2}}{1 - \kappa(s)q_1}$$

and consequently

$$|(\nabla \bar{X}(s, q_1, q_2))^{-1}| \leq \frac{\sqrt{9 + (\tau_0/\kappa_0)^2}}{1 - \kappa_0 r_0}$$

for all $(s, q_1, q_2) \in [s_1, s_2] \times D(r_0)$. The Lemma now follows from the global inverse function theorem (Appendix A.4).

From now on we assume that (2.62) is fulfilled. We introduce polar coordinates (θ, r) in $D(r_0)$ so that: $q_1 = r \cos \theta$, $q_2 = r \sin \theta$, $\theta \in [-\pi, \pi[$, $r \in]0, r_0[$. Then we may write

$$\begin{aligned} \bar{X}(r_0) &= \{X \in E \mid X = \bar{X}(s, \theta, r) = \hat{X}(s) + (\bar{p}(s) \cos \theta + \bar{b}(s) \sin \theta) r, \\ &\quad s \in [s_1, s_2], \theta \in [-\pi, \pi[, r \in]0, r_0[\} \end{aligned}$$

A tubular cross-section $T_{\bar{X}}(s, r_0)$ at the point s of $\bar{E}$ is defined by

$$T_{\bar{X}}(s, r_0) = \{X \in E \mid X = \bar{X}(s, \theta, r), \theta \in [-\pi, \pi[, r \in]0, r_0[\}$$

$s \in [s_1, s_2]$.

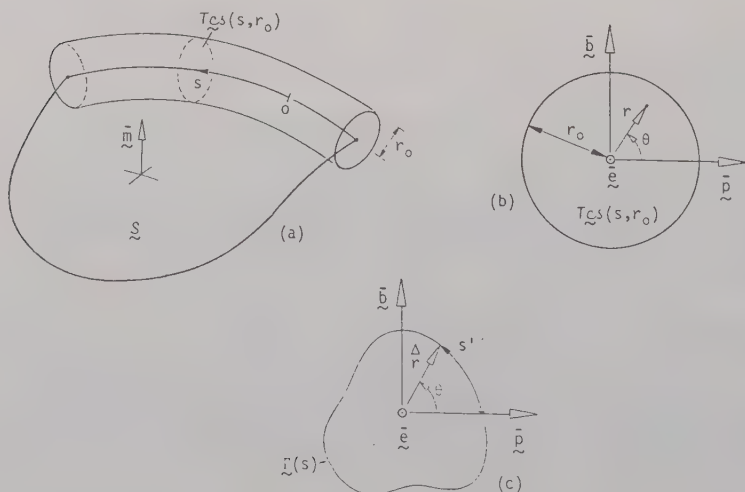


Figure 2.14

Let $\tilde{\Gamma}(s)$ be a simple closed curve in $T_{\tilde{\mathcal{C}}}^{\Delta}(s, r_0)$. We assume that $\tilde{\Gamma}(s)$, defined by

$$r = \tilde{r}^{\Delta}(s, \theta), \quad \theta \in [-\pi, \pi[\quad (2.63)$$

encircles the crack-front (see Figure 2.14(c)) and that the function $\tilde{r}^{\Delta}: [s_1, s_2] \times [-\pi, \pi[\rightarrow \mathbb{R}^+$ is smooth. A tubular surface $\tilde{T}_{\tilde{\mathcal{C}}}^{\Delta}$ in $T_{\tilde{\mathcal{N}}}^{\Delta}(r_0)$ is then defined by

$$\begin{aligned} \tilde{T}_{\tilde{\mathcal{C}}}^{\Delta} = \{ X \in E \mid X = \hat{X}(s, \theta) \equiv \hat{X}(s) + (\tilde{p}(s) \cos \theta + \tilde{b}(s) \sin \theta) \tilde{r}^{\Delta}(s, \theta), \\ s \in [s_1, s_2], \quad \theta \in [-\pi, \pi[\} \end{aligned} \quad (2.64)$$

The tubular surface is, according to our construction, generated by the family of plane curves $\tilde{\Gamma}(s)$, $s \in [s_1, s_2]$. We call $\tilde{\Gamma}(s)$ the cross-sectional curve of $\tilde{T}_{\tilde{\mathcal{C}}}^{\Delta}$ at the point s on $\tilde{E}$.

The area factor $\tilde{A} = \tilde{A}(s, \theta)$ of $\tilde{T}_{\tilde{\mathcal{C}}}^{\Delta}$, corresponding to the coordinates s and θ , is defined by

$$\tilde{A} = \left| \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial \theta} \right|$$

By a straightforward calculation we get

$$\tilde{A} = \sqrt{\left(\frac{\partial \tilde{r}}{\partial s} - \tilde{\tau} \frac{\partial \tilde{r}}{\partial \theta} \right)^2 \tilde{r}^2 + (1 - \tilde{\kappa} \tilde{r} \cos \theta)^2 \left(\left(\frac{\partial \tilde{r}}{\partial \theta} \right)^2 + \tilde{r}^2 \right)} \quad (2.65)$$

Note that $\tilde{A}(s, \theta) > 0$ for all $(s, \theta) \in [s_1, s_2] \times [-\pi, \pi[$. If we, instead of θ , use the arc-length coordinate s' on $\tilde{\Gamma}(s)$ (see Figure 2.14(c)) then the defining equation (2.63) may be written

$$r = \tilde{r}^{\Delta}(s, s'), \quad s' \in [s'_1(s), s'_2(s)[$$

(For the sake of simplicity we have used the same functional symbol $\tilde{r}^{\Delta}$ as in (2.63). It will always be clear from the context if θ or s' is used as coordinate) and the tubular surface is now represented by

$$\begin{aligned} \tilde{T}_{\tilde{\mathcal{C}}}^{\Delta} = \{ X \in E \mid X = \hat{X}(s, s') \equiv \hat{X}(s) + (\tilde{p}(s) \cos \theta(s, s') + \\ \tilde{b}(s) \sin \theta(s, s')) \tilde{r}^{\Delta}(s, s'), \quad s \in [s_1, s_2], \quad s' \in [s'_1(s), s'_2(s)[\} \end{aligned}$$

where

$$\theta(s, s') = -\pi + \int_{s_1(s)}^{s'} \frac{1}{\Delta} \sqrt{1 - \left(\frac{\partial r}{\partial s}\right)^2} ds'$$

and the corresponding area factor

$$\tilde{A} = \tilde{A}(s, s') = \left| \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial s'} \right|$$

is given by

$$\tilde{A} = \sqrt{\left(\frac{\partial r}{\partial s}\right)^2 \left[1 - \left(\frac{\partial r}{\partial s'}\right)^2\right] - \left(\tilde{\kappa} r \frac{\partial r}{\partial s'}\right)^2 + (1 - \tilde{\kappa} r \cos \theta)^2} \quad (2.66)$$

We introduce the following notations:

$$\text{rad}(T_s) = \sup_{\substack{\theta \in [-\pi, \pi[\\ s \in [s_1, s_2]}} \tilde{r}(s, \theta)$$

$$\text{int}(T_s) = \{X \in T_N(r_0), X \sim (s, \theta, r) \mid s \in [s_1, s_2], \theta \in [-\pi, \pi[, \\ r < \tilde{r}(s, \theta)\} \quad (2.67)$$

$$\text{int}(\tilde{\Gamma}(s)) = \{X \in T_{\tilde{\mathcal{A}}}(s, r_0), X \sim (s, \theta, r) \mid \theta \in [-\pi, \pi[, r < \tilde{r}(s, \theta)\}$$

A particularly simple tubular surface is obtained if we take

$$\tilde{r}(s, \theta) = R, \quad s \in [s_1, s_2], \quad \theta \in [-\pi, \pi[\quad (2.68)$$

where R is a real number: $0 < R < r_0$. We call this a circular tube with radius R and its area factor may be written

$$\tilde{A}(s, \theta) = |1 - \tilde{\kappa}(s) R \cos \theta| R \quad (2.69)$$

If the crack-front is a straight line ($\kappa(s) \equiv 0$) then (2.69) is reduced to

$$\tilde{A}(s, \theta) = R$$

We reinstall the time parameter by considering a moving circular tube

$\tilde{T}u_t(R)$, $t \in T$ accompanying the crack-front

$$\tilde{T}u_t(R) \equiv \{ X \in E \mid X = \hat{X}(s, t) + \tilde{e}_r(s, \theta, t) R \} \quad (2.70)$$

where

$$\tilde{e}_r(s, \theta, t) \equiv \tilde{p}_t(s) \cos \theta + \tilde{b}_t(s) \sin \theta$$

is the outward pointing (relative to $\text{int}(\tilde{T}u_t(R))$) unit normal vector on $\tilde{T}u_t(R)$. The normal velocity of the moving surface $\tilde{T}u_t(R)$, $t \in T$ is given by

$$\tilde{v}_t = \tilde{e}_r(t)(\tilde{e}_r(t) \cdot \tilde{c}_t) \quad (2.71)$$

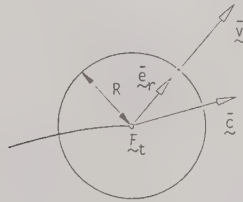


Figure 2.15

At each fixed time $t \in T$ we may extend the fields $\tilde{c}_t = \tilde{c}_t(X)$ and $\tilde{v}_t = \tilde{v}_t(X)$, defined on the crack-front, to fields defined on the tubular neighbourhood. We simply put

$$\begin{aligned} \tilde{c}_t(X) &\equiv \tilde{c}_t(\hat{X}(s, t)) & \tilde{T}N_t(r_0) \ni X = \hat{X}(s, \theta, r, t) \\ \tilde{v}_t(X) &\equiv \tilde{v}_t(\hat{X}(s, t)) \end{aligned} \quad (2.72)$$

and then $\tilde{c}_t(X) = \tilde{v}_t(X) \tilde{c}_t(X)$, $X \in \tilde{T}N_t(r_0)$. This may of course be done for any field defined on the crack-front.

Let Φ be a C^1 -fracture-field. Following Strifors [2] we introduce the convected crack-front time-derivative in the referential description

$$\Phi'(X, t) \equiv \frac{d}{dq} \Phi(X + q \tilde{c}_t(X), t + q) \big|_{q=0}, \quad X \in \tilde{T}N_t(r_0) \setminus \tilde{s}_t \quad (2.73)$$

where

$$0 < r_0 < \left(\sup_{s \in [s_1(t), s_2(t)]} \kappa_t(s) \right)^{-1}$$

By computing we get

$$\Phi' = \dot{\Phi} + (\text{Grad } \Phi) \bar{\zeta} = \dot{\Phi} + \zeta (\text{Grad } \Phi) \bar{\zeta} \quad (2.74)$$

We may also write

$$\Phi' = \frac{\partial}{\partial t} \Phi + (\text{grad } \Phi) x' \quad (2.75)$$

where

$$x' = \dot{x} + \underline{F} \bar{\zeta}, \quad (x' \equiv x') \quad (2.76)$$

Note that Φ' is a proper derivative in the sense that if Φ_1 and Φ_2 are scalar-fields and a, b are real numbers then: $(a\Phi_1 + b\Phi_2)' = a\Phi_1' + b\Phi_2'$, $(\Phi_1\Phi_2)' = \Phi_1'\Phi_2 + \Phi_1\Phi_2'$.

Lemma 2.8

$$J' = J(\text{div } x' - \text{Div } \bar{\zeta}) \quad (2.77)$$

Proof: From (2.74) and (2.18)_{1,2} in Lemma 2.3 we get

$$\begin{aligned} J' &= \dot{J} + \bar{\zeta} \cdot \text{Grad } J = J \text{div } \dot{x} + \bar{\zeta} \cdot J \text{div } \underline{F}^T = J(\text{div } \dot{x} + \text{div}(\underline{F} \bar{\zeta}) - \\ &\text{tr}(\underline{F} \text{grad } \bar{\zeta})) = J(\text{div } x' - \text{Div } \bar{\zeta}) \end{aligned}$$

□

Remark: By a reasonably straightforward calculation we get

$$\text{Div } \bar{\zeta} = \zeta \text{Div } \bar{\zeta} = \zeta \frac{\kappa_g}{1 - \kappa r \cos \theta} \quad (2.78)$$

where $\kappa_g = \kappa_{g,t}(s) \equiv \kappa_t(s) \sin \varphi_t(s)$ is the geodesic curvature of $\underline{F}_t$ on Σ_t . (See Appendix (A.27)).

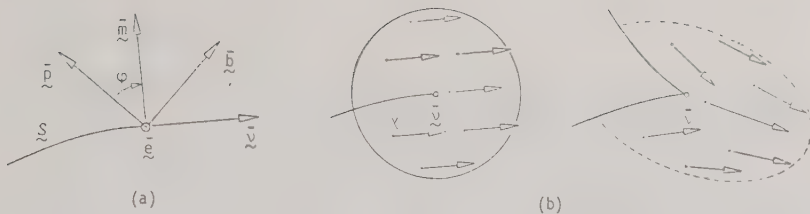


Figure 2.16

We introduce the field $\bar{v} = \bar{v}_t(X)$ defined by

$$\bar{v}_t(X) \equiv E(X, t) \bar{\omega}_t(X), \quad X \in TN_t(r_0) \quad (2.79)$$

(see the illustration in Figure 2.16 (b) above). We may then write

$$x' = \dot{x} + \bar{v} \bar{c} \quad (2.80)$$

The Piola-fields corresponding to $\Phi' + \Phi \operatorname{div} x'$, $\operatorname{div}(\Phi \otimes \bar{v})$ and $\operatorname{div}(\Phi \otimes (x' - \dot{x}))$ are given by

Lemma 2.9

$$\begin{aligned} \Phi_p' + \Phi_p \operatorname{Div} \bar{\omega} &= J(\Phi' + \Phi \operatorname{div} x') \\ \operatorname{Div}(\Phi_p \otimes \bar{\omega}) &= J \operatorname{div}(\Phi \otimes \bar{v}) \\ \operatorname{Div}(\Phi_p \otimes \bar{\omega}) &= J \operatorname{div}(\Phi \otimes (x' - \dot{x})) \end{aligned} \quad (2.81)$$

Proof: We have $\Phi_p = J\Phi$ and then $\Phi_p' = J'\Phi + J\Phi'$. Hence by Lemma 2.8

$$\Phi_p' = J(\Phi' + \Phi \operatorname{div} x' - \Phi \operatorname{Div} \bar{\omega})$$

and this proves (2.81)₁. Formula (2.81)₂ follows from

$$\begin{aligned} \operatorname{Div}(\Phi_p \otimes \bar{\omega}) &= (\operatorname{Grad} \Phi_p) \bar{\omega} + \Phi_p \operatorname{Div} \bar{\omega} = J((\operatorname{grad} \Phi) E + \Phi \otimes \operatorname{div} F^T) \bar{\omega} + \\ \Phi_p \operatorname{Div} \bar{\omega} &= J((\operatorname{grad} \Phi) \bar{v} + \Phi(\operatorname{div} \bar{v} - \operatorname{Div} \bar{\omega}) + \Phi \operatorname{Div} \bar{\omega}) = J((\operatorname{grad} \Phi) \bar{v} + \end{aligned}$$

$$+ \Phi \operatorname{div} \vec{v}) = J \operatorname{div}(\Phi \otimes \vec{v})$$

where we have used (2.47)₂. Formula (2.81)₃ is a simple consequence of (2.81)₂ and (2.80). Note that $\vec{v} \cdot \operatorname{Grad} \zeta = 0$. $\square$

We now return to the problem concerning the calculation of the derivative of the function (2.48).

Proposition 2.9 Let P be a subbody of B , i.e. a fixed regular sub-region of B . For the sake of simplicity we assume that ∂P is smooth. Let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations (satisfying K1 - K8) and consider the mapping

$$\chi^\partial(\cdot, t) : \partial P \setminus \tilde{S}_t \rightarrow \partial P_t^+ \equiv \partial P_t \setminus (p_t^+ \cup p_t^-)$$

where χ^∂ denotes the restriction of χ to $\partial P \setminus \tilde{S}_t$ and $p_t^\pm \equiv \chi^\pm(\tilde{S}_t \cap P, t)$. Let $A^\partial = A^\partial(\chi, t)$ denote the area-ratio corresponding to χ^∂ .

We assume that

$$\text{K10: } A^\partial \in L^1(\partial P \setminus \tilde{S}_t)$$

If in addition

- a) $P \subset \tilde{N}_t(r_0) \cup \tilde{F}_t$ for all $t \in T^-$, where T^- is an open subinterval of T and

$$0 < r_0 < \sup_{t \in T^-} \left(\sup_{s \in [s_1(t), s_2(t)]} \kappa_t(s) \right)^{-1}$$

- b) $\tilde{S}_t \cap P$ is a plane surface for all $t \in T^-$.

and Φ is a C^1 -fracture-density satisfying

- c) $\Phi' + \Phi \operatorname{div} x' \in L^1(P_t)$ uniformly in time ($t \in T^-$).
- d) $\Phi(x' - \dot{x}) \cdot \bar{n} \in L^1(\partial P_t^+)$ uniformly in time ($t \in T^-$), where $\bar{n} = \bar{n}_t(x)$ is the outward unit normal on ∂P_t^+ .

then

$$T^- \ni t \mapsto \int_{P_t} \Phi(x', t) \, dv(x)$$

is a C^1 - function and

$$\frac{d}{dt} \int_{P_t} \Phi \, dv(x) = \int_{P_t} \{\Phi' + \Phi \operatorname{div} x'\} \, dv(x) - \int_{\partial P_t^+} \Phi (x' - \dot{x}) \cdot \bar{n} \, da(x) \quad (2.82)$$

Proof: (cf. Gurtin & Yatomi [9], Lemma 2). Let R_δ , $\delta > 0$ be a family of open rectangular boxes in E with two boundary-surfaces parallel to the crack-surface in the referential description ($\tilde{S}_t \cap P$, $t \in T^+$) and with the mutual distance δ .

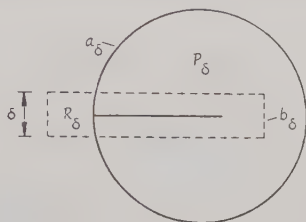


Figure 2.17

We assume that for every $\delta > 0$: $\tilde{S}_t \cap P \subset R_\delta$ for all $t \in T^+$ and we have

$\operatorname{volume}(R_\delta) \rightarrow 0$ and $\operatorname{area}(\partial P \cap R_\delta) \rightarrow 0$ as $\delta \rightarrow 0$

Put $P_\delta \equiv P \setminus \bar{R}_\delta$ and $a_\delta \equiv \partial P \setminus (\partial P \cap \bar{R}_\delta)$, then $P_{t,\delta} \equiv X(P_\delta, t)$ and $a_{t,\delta} \equiv X(a_\delta, t)$ are open sets in P_t and on the surface ∂P_t^+ respectively and

$\operatorname{volume}(P_t \setminus P_{t,\delta}) \rightarrow 0$ and $\operatorname{area}(\partial P_t^+ \setminus a_{t,\delta}) \rightarrow 0$ as $\delta \rightarrow 0$

We have ($t \in T^+$)

$$\begin{aligned} \frac{d}{dt} \int_{P_t} \Phi \, dv(x) &= \frac{d}{dt} \int_{P_\delta} \Phi_P \, dv(X) = \int_{P_\delta} \dot{\Phi}_P \, dv(X) = \int_{P_\delta} (\Phi'_P - (\operatorname{Grad} \Phi_P) \cdot \bar{\zeta}) \, dv(X) = \\ &= \int_{P_\delta} (\Phi'_P + \Phi_P \operatorname{Div} \bar{\zeta}) \, dv(X) - \int_{P_\delta} \operatorname{Div}(\Phi_P \otimes \bar{\zeta}) \, dv(X) \end{aligned} \quad (2.83)$$

Let b_δ denote the "front-surface" of the region P_δ as shown in

Figure 2.17. From the divergence theorem we get

$$\int_{P_\delta} \text{Div}(\Phi_p \otimes \tilde{c}) \, dv(X) = \int_{a_\delta} \Phi_p \tilde{c} \cdot \tilde{n} \, da(X) - \int_{b_\delta} \Phi_p \tilde{c} \, da(X) \quad (2.84)$$

where $\tilde{n} = \tilde{n}(X)$ denotes the outward unit normal on ∂P . From the geometric relation (see section 3.2, Lemma 3.1)

$$\tilde{n} = (A^\partial)^{-1} J F^{-T} \tilde{n}$$

where $\tilde{n} = \tilde{n}_t(x)$ denotes the unit normal on ∂P_t^+ (at the point $x = X(X, t)$), and (2.80) we obtain

$$\tilde{c} \cdot \tilde{n} = A^\partial J^{-1} (x' - \dot{x}) \cdot \tilde{n} \quad (2.85)$$

From (2.83) - (2.85) and (2.81)₁ we get

$$\begin{aligned} \frac{d}{dt} \int_{P_{t,\delta}} \Phi \, dv(x) &= \int_{P_{t,\delta}} (\Phi' + \Phi \, \text{div} \, x') \, dv(x) - \int_{a_{t,\delta}} \Phi (x' - \dot{x}) \cdot \tilde{n} \, da(x) - \\ &\int_{b_\delta} \Phi_p \tilde{c} \, da(X) \end{aligned} \quad (2.86)$$

where

$$\int_{b_\delta} \Phi_p \tilde{c} \, da(X) \rightarrow 0 \text{ uniformly in } T^- \text{ as } \delta \rightarrow 0 \quad (2.87)$$

(note that $\text{area}(b_\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and that $\Phi_p(\cdot, t)$ is regular on b_δ for all $t \in T^-$). From (2.86), (2.87), c) and d) we conclude that

$$\frac{d}{dt} \int_{P_{t,\delta}} \Phi \, dv(x) \rightarrow \int_{P_t} (\Phi' + \Phi \, \text{div} \, x') \, dv(x) - \int_{\partial P_t^+} \Phi (x' - \dot{x}) \cdot \tilde{n} \, da(x)$$

uniformly in T^- as $\delta \rightarrow 0$. But

$$\int_{P_{t,\delta}} \Phi \, dv(x) \rightarrow \int_{P_t} \Phi \, dv(x) \text{ pointwise in } T^- \text{ as } \delta \rightarrow 0$$

and this proves the Proposition. $\square$

Remark 1: Note that since

$$\int_{P_t} (\Phi)^\pm (x' - \dot{x})^\pm \cdot \bar{m}^\pm da(x) = \mp \int_{S_t} \int_P (\Phi_P)^\pm \bar{c} \cdot \bar{m} da(X) = 0$$

we may write

$$\int_{\partial P_t^+} \Phi (x' - \dot{x}) \cdot \bar{n} da(x) = \int_{\partial P_t} \Phi (x' - \dot{x}) \cdot \bar{n} da(x) \quad (2.88)$$

Remark 2: In the referential description formula (2.82) reads

$$\frac{d}{dt} \int_P \Phi_P dv(x) = \int_P (\Phi_P' + \Phi_P \operatorname{Div} \bar{c}) dv(X) - \int_{\partial P} \Phi_P \bar{c} \cdot \bar{n} da(X) \quad (2.89)$$

Corollary 2.3 Let P be a subbody of B . If, in addition to the assumptions stated in Proposition 2.9, we have

$$e) \quad \bar{F}_t \subset \bar{P} \quad \text{for all } t \in T^-$$

then

$$T^- \ni t \mapsto \int_{B_t} \Phi(x, t) dv(x)$$

is a C^1 -function and

$$\begin{aligned} \frac{d}{dt} \int_{B_t} \Phi dv(x) &= \int_{B_t \setminus P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x}) dv(x) + \int_{P_t} (\Phi' + \Phi \operatorname{div} x') dv(x) - \\ &\quad \int_{\partial P_t} \Phi (x' - \dot{x}) \cdot \bar{n} da(x) \end{aligned} \quad (2.90)$$

Proof: We have ($t \in T^-$)

$$\frac{d}{dt} \int_{B_t} \Phi dv(x) = \frac{d}{dt} \int_{B_t \setminus P_t} \Phi dv(x) + \frac{d}{dt} \int_{P_t} \Phi dv(x)$$

and since X and Φ are regular on $(B \setminus P) \times T^-$

$$\frac{d}{dt} \int_{B_t \setminus P_t} \Phi dv(x) = \int_{B_t \setminus P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x}) dv(x)$$

The Corollary now follows from (2.82). $\square$

Remark: Note that

$$\dot{\Phi} + \Phi \operatorname{div} \dot{X} = \Phi' + \Phi \operatorname{div} X' - \operatorname{div}(\Phi \otimes (X' - \dot{X})) \quad (2.91)$$

and thus, if the divergence theorem may be applied to the surface integral in (2.90), i.e.

$$\int_{\partial P_t} \Phi(X' - \dot{X}) \cdot \bar{n} \, da(x) = \int_{P_t} \operatorname{div}(\Phi \otimes (X' - \dot{X})) \, dv(x) \quad (2.92)$$

then formula (2.90) can be brought back to (2.50). However, due to the possible singularity at the crack-front, formula (2.92) can not be said to be generally valid.

The introduction of the convected crack-front time-derivative (2.73) is accompanied by the conjecture that Φ' will be less singular at the crack-front than $\dot{\Phi}$ and $\operatorname{Grad} \Phi$. This conjecture is motivated by the very construction of Φ' as a time-derivative performed in a "frame" accompanying the motion of the crack-front in the referential description:

$$\Phi'(X, t) = \lim_{q \rightarrow 0} \frac{\Phi(X + q \tilde{C}_t(X), t + q) - \Phi(X, t)}{q}$$

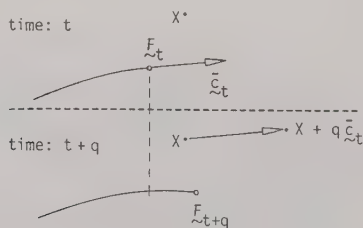


Figure 2.18

The condition $\Phi' = 0$ reflects the stationary character of Φ as seen from this "frame". If Φ' is bounded at the crack-front and $\operatorname{Grad} \Phi$ is unbounded then asymptotically (as $\tilde{F}$ is approached from $B \setminus \tilde{S}$) we have

$$\dot{\Phi} \simeq -(\operatorname{Grad} \Phi) \tilde{C} \quad (2.93)$$

For instance, if X' is bounded and $\tilde{F}$ is unbounded then

$$\dot{\mathbf{x}} \approx -\bar{\mathbf{v}} \underline{\mathbf{c}} \quad (2.94)$$

Remark: These matters are discussed in Strifors [2], chapter 2.

Formula (2.82) depends on the underlying reference configuration not only through the condition that $\underline{\mathcal{S}}_t \cap P$, $t \in T^-$ are plane surfaces but also through the fact that the convected crack-front time-derivative depends on the reference configuration.

A change of reference configuration is represented by a diffeomorphism

$$\lambda: B = B_0 \rightarrow B_1, \quad (\lambda(X) = X_1) \quad (2.95)$$

With $\underline{\Pi} = \underline{\Pi}(X, t) \equiv \nabla \lambda(X, t)$ ($\det \underline{\Pi} > 0$) we have

$$\begin{aligned} \underline{F}(X, t) &= \underline{F}_1(X_1, t) \underline{\Pi}(X) \\ (\text{Grad } \Phi)(X, t) &= (\text{Grad}_1 \Phi)(X_1, t) \underline{\Pi}(X)^{-1} \end{aligned} \quad (2.96)$$

$X_1 = \lambda(X)$, $X \in B \setminus \underline{\mathcal{S}}_t$, where $\underline{F}_1$ denotes the deformation gradient relative to the reference configuration B_1 and $(\text{Grad}_1 \Phi)(X_1, t) \equiv (\partial/\partial X_1) \Phi(X_1, t)$. The crack-surface and the crack-front are, in the reference configuration B_1 , represented by (the time parameter is omitted)

$$\underline{\mathcal{S}}_1 = \lambda(\underline{\mathcal{S}}), \quad \underline{\mathcal{F}}_1 = \lambda(\underline{\mathcal{F}})$$

Let $\bar{\underline{\mathcal{C}}}_1 = \bar{\underline{\mathcal{V}}}_1 \underline{\mathcal{C}}_1$ denote the crack-front velocity in the reference configuration B_1 . Then

Lemma 2.10

$$\begin{aligned} \bar{\underline{\mathcal{V}}}_1(Z_1) &= L_\lambda(Z) (A_\lambda(Z))^{-1} \underline{\Pi}(Z) (\bar{\underline{\mathcal{V}}}(Z) - \frac{\underline{\Pi}(Z) \bar{\underline{\mathcal{E}}}(Z) \cdot \underline{\Pi}(Z) \bar{\underline{\mathcal{V}}}(Z)}{(L_\lambda(Z))^2} \bar{\underline{\mathcal{E}}}(Z)) \\ \underline{\mathcal{C}}_1(Z_1) &= \underline{\mathcal{C}}(Z) (L_\lambda(Z))^{-1} A_\lambda(Z) \end{aligned} \quad (2.97)$$

$Z_1 = \lambda(Z)$, $Z \in \underline{\mathcal{F}}$, where L_λ is the length ratio associated with the restriction of λ to $\underline{\mathcal{F}}$ and A_λ is the area ratio associated with the restriction of λ to $\underline{\mathcal{S}}$.

Proof: Let $\bar{\underline{\mathcal{E}}}_1 = \bar{\underline{\mathcal{E}}}_1(Z_1)$ denote the unit tangent vector field on $\underline{\mathcal{F}}_1$ satisfying

¹ Φ is vector-valued

$$\Pi \bar{\tilde{e}} = L_\lambda \bar{\tilde{e}}_1 \quad (2.98)$$

With $\Pi_S \equiv \nabla_\Delta \lambda_S$, where λ_S denotes the restriction of λ to $\underline{S}$ (and ∇_Δ is the surface gradient on $\underline{S}$), we have (cf. Lemma 3.3)

$$\bar{\tilde{v}}_1 = L_\lambda^{-1} A_\lambda \Pi_S^{-T} \bar{\tilde{v}} \quad (2.99)$$

From (2.98) and (2.99) we obtain

$$\bar{\tilde{e}}_1 \cdot \Pi \bar{\tilde{v}} = \frac{\Pi \bar{\tilde{e}} \cdot \Pi \bar{\tilde{v}}}{L_\lambda}, \quad \bar{\tilde{v}}_1 \cdot \Pi \bar{\tilde{v}} = L_\lambda^{-1} A_\lambda \Pi_S^{-T} \bar{\tilde{v}} \cdot \Pi_S \bar{\tilde{v}} = L_\lambda^{-1} A_\lambda$$

and consequently

$$\Pi \bar{\tilde{v}} = \frac{\Pi \bar{\tilde{e}} \cdot \Pi \bar{\tilde{v}}}{L_\lambda} \bar{\tilde{e}}_1 + L_\lambda^{-1} A_\lambda \bar{\tilde{v}}_1 \quad (2.100)$$

Formula (2.97)₁ now follows from (2.98) and a simple re-organization of (2.100).

If $\underline{F}_t = \{X \in E \mid X = \hat{X}(q, t), q \in [a, b]\}$ then $\underline{F}_{1t} = \{X_1 \in E \mid X_1 = \lambda(\hat{X}(q, t)), q \in [a, b]\}$ and

$$\bar{\tilde{c}}_1 = \Pi \frac{\partial \hat{X}}{\partial t} - (\bar{\tilde{e}}_1 \cdot \Pi \frac{\partial \hat{X}}{\partial t}) \bar{\tilde{e}}_1 \quad (2.101)$$

where

$$\frac{\partial \hat{X}}{\partial t} = \bar{\tilde{c}} + (\bar{\tilde{e}} \cdot \frac{\partial \hat{X}}{\partial t}) \bar{\tilde{e}} \quad (2.102)$$

From (2.98), (2.101), (2.102) and (2.97)₁ we get

$$\bar{\tilde{c}}_1 = \Pi \bar{\tilde{c}} - (\bar{\tilde{e}}_1 \cdot \Pi \bar{\tilde{c}}) \bar{\tilde{e}}_1 = \underline{c} \Pi(\bar{\tilde{v}} - \frac{\Pi \bar{\tilde{e}} \cdot \Pi \bar{\tilde{v}}}{L_\lambda^2} \bar{\tilde{e}}) = \bar{\tilde{v}}_1 \underline{c} L_\lambda^{-1} A_\lambda$$

and this proves (2.97)₂. $\square$

We introduce the field $\bar{g} = \bar{g}_t(X)$ defined by

$$\bar{g}_t(X) \equiv \underline{F}(X, t) \bar{\tilde{e}}_t(X), \quad X \in \mathcal{N}_t(r_0) \quad (2.103)$$

Let $\bar{g}_1$ denote the corresponding field defined relative to the reference

configuration B_1 . Furthermore let Φ'_1 denote the convected crack-front time derivative defined relative to the reference configuration B_1 and let, analogous to the definition (2.79), $\bar{v}_1 \equiv \underline{F}_1 \bar{v}_1$.

Let $T_{\mathcal{CS}}(Z, r_0)$ and $T_{\mathcal{CS}_1}(Z_1, r_1)$ denote tubular cross-sections at points $Z \in \underline{F}$ and $Z_1 \in \underline{F}_1$ respectively. We introduce the field $A_{\lambda, F} = A_{\lambda, F}(X)$, $X \in \underline{T\mathcal{N}}(r_0)$ defined by

$$A_{\lambda, F}(X) \equiv A_{\lambda}(Z), \quad X \in T_{\mathcal{CS}}(Z, r_0) \quad (2.104)$$

Proposition 2.10 If λ is a change of reference configuration satisfying:

$$a) \quad X \in T_{\mathcal{CS}}(Z, r_0) \Rightarrow \lambda(X) \in T_{\mathcal{CS}_1}(\lambda(Z), r_1)$$

$$b) \quad X \in T_{\mathcal{CS}}(Z, r_0) \Rightarrow \underline{\Pi}(X) \bar{\underline{e}}(X) = \underline{\Pi}(Z) \bar{\underline{e}}(Z) \text{ and } \underline{\Pi}(X) \bar{\underline{v}}(X) = \underline{\Pi}(Z) \bar{\underline{v}}(Z)$$

for every $Z \in \underline{F}$, then

$$\bar{g}_1(X_1) = (L_{\lambda}(X))^{-1} \bar{g}(X)$$

$$\bar{v}_1(X_1) = L_{\lambda}(X) (A_{\lambda, F}(X))^{-1} \bar{v}(X) \quad (2.105)$$

$$\Phi'_1(X_1) = \Phi'(X)$$

$X_1 = \lambda(X)$, $X \in \underline{T\mathcal{N}}(r_0)$ (r_0 sufficiently small).

Remark: The reader is reminded of the definitions

$$\bar{\underline{e}}(X) \equiv \bar{\underline{e}}(Z), \quad \bar{\underline{v}}(X) \equiv \bar{\underline{v}}(Z), \quad X \in T_{\mathcal{CS}}(Z, r_0)$$

Proof: If $X \in T_{\mathcal{CS}}(Z, r_0)$, $X_1 = \lambda(X)$ then

$$\bar{g}_1(X_1) = \underline{F}_1(X_1) \bar{\underline{e}}_1(X_1) = \underline{F}(X) (\underline{\Pi}(X))^{-1} (L_{\lambda}(Z))^{-1} \underline{\Pi}(Z) \bar{\underline{e}}(Z) = (L_{\lambda}(X))^{-1} \bar{g}(X)$$

where we have used (2.96)₁ and b). This proves (2.105)₁. From a) it follows that if $Z \in \underline{F}$, $Z_1 = \lambda(Z)$ then

$$0 = \bar{\underline{e}}_1(Z_1) \cdot (\lambda(Z + q \bar{v}(Z)) - \lambda(Z)) = \bar{\underline{e}}_1(Z_1) \cdot q \underline{\Pi}(Z) \bar{\underline{v}}(Z) + \bar{\underline{e}}_1(Z_1) \cdot \bar{o}(q)$$

where $\frac{1}{q} \bar{o}(q) \rightarrow \bar{o}$ as $q \rightarrow 0$. Consequently

$$\bar{\underline{e}}_1(Z_1) \cdot \underline{\Pi}(Z) \bar{\underline{v}}(Z) = 0$$

and from (2.97)₁ we may conclude that

$$\bar{\varphi}_1(Z_1) = L_\lambda(Z)(A_\lambda(Z))^{-1} \bar{\Pi}(Z) \bar{\varphi}(Z) \quad (2.106)$$

If $X \in T_{\mathcal{CS}}(Z, r_0)$, $X_1 = \lambda(X)$ then

$$\begin{aligned} \bar{\varphi}_1(X_1) &= \bar{F}_1(X_1) \bar{\varphi}_1(X_1) = \bar{F}(X)(\bar{\Pi}(X))^{-1} L_\lambda(Z)(A_\lambda(Z))^{-1} \bar{\Pi}(Z) \bar{\varphi}(Z) = \\ &= L_\lambda(X)(A_{\lambda, F}(X))^{-1} \bar{\varphi}(X) \end{aligned}$$

where we have used (2.96)₁, (2.106) and b). This proves (2.105)₂.

Finally we have

$$\begin{aligned} \Phi'_1(X_1) &= \dot{\Phi}(X_1) + \underline{c}_1(X_1)(\text{Grad}_1 \Phi)(X_1) \bar{\varphi}_1(X_1) = \\ &= \dot{\Phi}(X) + \underline{c}(X)(L_\lambda(X))^{-1} A_{\lambda, F}(X)(\text{Grad} \Phi)(X)(\bar{\Pi}(X))^{-1} L_\lambda(Z)(A_\lambda(Z))^{-1} \bar{\Pi}(Z) \bar{\varphi}(Z) = \\ &= \dot{\Phi}(X) + \underline{c}(X)(\text{Grad} \Phi)(X) \bar{\varphi}(X) = \Phi'(X) \end{aligned} \quad (2.107)$$

where we have used (2.96)₂, (2.97)₂ and (2.106). $\square$

An arbitrary change of reference configuration will mix the material points among the tubular cross-sections, that is, if $X, Y \in T_{\mathcal{CS}}(Z, r_0)$ then, in general, $\lambda(X) \in T_{\mathcal{CS}_1}(Z_1, r_1)$, $\lambda(Y) \in T_{\mathcal{CS}_1}(Z'_1, r_1)$ where $Z_1 \neq Z'_1$ ($\neq \lambda(Z)$). In this case there is no simple relation between Φ' and Φ'_1 ! A change of reference configuration satisfying condition a) in Proposition 2.10 is called simple. We introduce the assumption

K11: There is a simple change of reference configuration $\lambda: B \rightarrow B_1$ with $\underline{S}_{1,t} = \lambda(\underline{S}_t)$ a plane surface and $F_{1,t} = \lambda(\underline{F}_t)$ a straight line for all $t \in T$.

Proposition 2.11 Let P be a subbody of B and let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying (K1-K8), K10 and K11.

Assume that

a) $P \subset T_{N_t}(r_0) \cup \underline{F}_t$ for all $t \in T^-$

and that Φ is a C^1 -fracture-density satisfying

b) $\text{div}(\Phi \otimes \bar{a}) \in L^1(P_t)$ and $\Phi \bar{a} \cdot \bar{n} \in L^1(\partial P_t^+)$, both uniformly in time ($t \in T^-$) and

$$\int_{\partial P_t} \Phi \bar{a} \cdot \bar{n} da(x) = \int_{P_t} \operatorname{div}(\Phi \otimes \bar{a}) dv(x) \quad (2.108)$$

for every C^1 -fracture-field $\bar{a}$ satisfying

$$\lim_{\substack{X \rightarrow Z \\ X \in T_{\mathcal{C}^1}(Z, r_0)}} |X - Z|^{-\alpha} (F^{-1} \bar{a})(X, t) = \bar{0}, \quad Z \in \tilde{F}_t \quad (2.109)$$

for all $\alpha < 1$.

c) $\Phi' + \Phi \operatorname{div} x' \in L^1(P_t)$ and $\Phi(x' - \dot{x}) \cdot \bar{n} \in L^1(\partial P_t^+)$, both uniformly in time ($t \in T^-$).

Then

$$T^- \ni t \mapsto \int_{P_t} \Phi(x, t) dv(x)$$

is a C^1 -function and

$$\frac{d}{dt} \int_{P_t} \Phi dv(x) = \int_{P_t} (\Phi' + \Phi \operatorname{div} x') dv(x) - \int_{\partial P_t} \Phi(x' - \dot{x}) \cdot \bar{n} da(x) \quad (2.110)$$

Proof: If λ is simple then (cf. (2.107))

$$\Phi'_1(X_1, t) = \Phi'(X, t) + (\operatorname{Grad} \Phi)(X, t) \bar{\pi}_t(X) \quad (2.111)$$

where $\bar{\pi} = \bar{\pi}_t(X)$, $X \in T_{N_t}(r_0)$ is defined by

$$\bar{\pi}_t(X) \equiv \mathcal{C}_t(X) ((\Pi(X))^{-1} \Pi(Z) - \mathbb{I}) \bar{\nu}_t(X), \quad X \in T_{\mathcal{C}^1}(Z, r_0) \quad (2.112)$$

It is a simple matter to show that

$$\bar{\pi}_t(X) = \mathcal{C}_t(X) ((\nabla \Pi^{-1})(Z)(X - Z)) \Pi(Z) \bar{\nu}_t(X) + o(|X - Z|) \quad (2.113)$$

where $|X - Z|^{-1} o(|X - Z|) \rightarrow 0$ as $X \rightarrow Z$. Consequently

$$\lim_{\substack{X \rightarrow Z \\ X \in T_{\mathcal{C}^1}(Z, r_0)}} |X - Z|^{-\alpha} \bar{\pi}_t(X) = \bar{0} \quad (2.114)$$

for all $\alpha < 1$.

From (2.111) we get the relations

$$\begin{aligned}\Phi'_1 + \Phi \operatorname{div} x'_1 &= \Phi' + \Phi \operatorname{div} x' + \operatorname{div}(\Phi \otimes \bar{\pi}) \\ \Phi(x'_1 - \dot{x}) \cdot \bar{n} &= \Phi(x' - \dot{x}) \cdot \bar{n} + \Phi \bar{\pi} \cdot \bar{n}\end{aligned}\quad (2.115)$$

where $\bar{\pi} = \underline{F} \bar{\pi}$, and thus, from (2.114) and b) (with $\bar{a} = \bar{\pi}$) we conclude that $\Phi'_1 + \Phi \operatorname{div} x'_1 \in L^1(P_t)$ and $\Phi(x'_1 - \dot{x}) \cdot \bar{n} \in L^1(\partial P_t^+)$, both uniformly in time ($t \in T^-$). Now take the reference configuration B_1 as in K11. Proposition 2.9 then gives

$$\frac{d}{dt} \int_{P_t} \Phi \, dv(x) = \int_{P_t} (\Phi'_1 + \Phi \operatorname{div} x'_1) \, dv(x) - \int_{\partial P_t} \Phi(x'_1 - \dot{x}) \cdot \bar{n} \, da(x) \quad (2.116)$$

Formula (2.110) now follows from (2.108), (2.115) and (2.116). $\square$

Remark 1: In the referential description (2.108) may be written

$$\int_{\partial P} \Phi_P \bar{a} \cdot \bar{n} \, da(X) - \int_{\tilde{S}_t \cap P} [\Phi_P \otimes \bar{a}] \bar{m} \, da(X) = \int_P \operatorname{Div}(\Phi_P \otimes \bar{a}) \, dv(X) \quad (2.117)$$

where $\bar{a} = \underline{F}^{-1} \bar{a}$ and $[\Phi] = (\Phi)^+ - (\Phi)^-$.

Remark 2: Note that (2.109) does not depend on the reference configuration.

Remark 3: The assumption made in K11, that $\underline{F}_{1,t}$, $t \in T$ are all straight lines, is not essential if we agree to take P sufficiently close to the crack-front.

Consider the crack-front in the spatial description F_t , $t \in T$. Let $TN_t(x)$ denote a tubular neighbourhood to F_t .

Let Φ be a C^1 -fracture-field. We introduce the convected crack-front time-derivative in the spatial description

$$\delta \Phi(x, t) = \frac{d}{dq} \Phi(x + q \bar{c}_t(x), t + q) \big|_{q=0}, \quad x \in TN_t(x) \cap \mathcal{B}_t \quad (2.118)$$

where $\bar{c}$ is the crack-front velocity in the spatial description and the field $\bar{c}_t = \bar{c}_t(x)$, $x \in TN_t(x)$ is defined in the obvious manner.

By computing we get

$$\delta \Phi = \frac{\partial}{\partial t} \Phi + (\operatorname{grad} \Phi) \bar{c} \quad (2.119)$$

and the simple relation

$$\delta \Phi = \Phi' + (\text{grad } \Phi)(\bar{c} - x') \quad (2.120)$$

Proposition 2.12 Let P be a subbody of B and let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying (K1-K8), K10 and K11.

Assume that

$$a) \quad P \subset \tilde{T}N_t(r_0) \cup F_t \quad \text{and} \quad P_t \subset TN_t(x) \cup F_t \quad \text{for all } t \in T^-.$$

and that Φ is a C^1 -fracture-density satisfying

$$b) \quad \text{in Proposition 2.11}$$

$$c) \quad \text{div}(\Phi \otimes (x' - \bar{c})) \in L^1(P_t) \quad \text{and} \quad \Phi(x' - \bar{c}) \cdot \bar{n} \in L^1(\partial P_t^+), \quad \text{both uniformly in time } (t \in T^-) \quad \text{and}$$

$$\int_{\partial P_t} \Phi(x' - \bar{c}) \cdot \bar{n} \, da(x) = \int_{P_t} \text{div}(\Phi \otimes (x' - \dot{x})) \, dv(x) \quad (2.121)$$

$$d) \quad \delta \Phi + \Phi \text{div } \bar{c} \in L^1(P_t) \quad \text{and} \quad \Phi(\bar{c} - \dot{x}) \cdot \bar{n} \in L^1(\partial P_t^+), \quad \text{both uniformly in time } (t \in T^-).$$

Then

$$T^- \ni t \mapsto \int_{P_t} \Phi(x, t) \, dv(x)$$

is a C^1 -function and

$$\frac{d}{dt} \int_{P_t} \Phi \, dv(x) = \int_{P_t} (\delta \Phi + \Phi \text{div } \bar{c}) \, dv(x) - \int_{\partial P_t} \Phi(\bar{c} - \dot{x}) \cdot \bar{n} \, da(x) \quad (2.122)$$

Proof: We have the relations

$$\begin{aligned} \Phi' + \Phi \text{div } x' &= \delta \Phi + \Phi \text{div } \bar{c} + \text{div}(\Phi \otimes (x' - \bar{c})) \\ \Phi(x' - \dot{x}) \cdot \bar{n} &= \Phi(x' - \bar{c}) \cdot \bar{n} + \Phi(\bar{c} - \dot{x}) \cdot \bar{n} \end{aligned} \quad (2.123)$$

and thus, from c) and d), we may conclude that $\Phi' + \Phi \text{div } x' \in L^1(P_t)$ and $\Phi(x' - \dot{x}) \cdot \bar{n} \in L^1(\partial P_t^+)$, both uniformly in time $(t \in T^-)$.

Proposition 2.11 and relation (2.123) then give

$$\begin{aligned} \frac{d}{dt} \int_{P_t} \Phi dv(x) &= \int_{P_t} (\delta \Phi + \Phi \operatorname{div} \bar{c} + \operatorname{div}(\Phi \otimes (x' - \bar{c}))) dv(x) - \\ &\int_{\partial P_t} (\Phi (x' - \bar{c}) \cdot \bar{n} + \Phi (\bar{c} - \dot{x}) \cdot \bar{n}) da(x) \end{aligned} \quad (2.124)$$

and formula (2.122) now follows from (2.124) and (2.121). $\square$

Remark 1: Note that

$$\delta \Phi + \Phi \operatorname{div} \bar{c} = \dot{\Phi} + \Phi \operatorname{div} \dot{x} + \operatorname{div}(\Phi \otimes (\bar{c} - \dot{x})) \quad (2.125)$$

and thus (2.122) may be written

$$\begin{aligned} \frac{d}{dt} \int_{P_t} \Phi dv(x) &= \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x} + \operatorname{div}(\Phi \otimes (\bar{c} - \dot{x}))) dv(x) - \\ &\int_{\partial P_t} \Phi (\bar{c} - \dot{x}) \cdot \bar{n} da(x) \end{aligned} \quad (2.126)$$

and if

$$\int_{\partial P_t} \Phi (\bar{c} - \dot{x}) \cdot \bar{n} da(x) = \int_{P_t} \operatorname{div}(\Phi \otimes (\bar{c} - \dot{x})) dv(x) \quad (2.127)$$

then we are once again brought back to formula (2.50).

Remark 2: Since $(\Phi)^\pm$ may be singular at the crack-front we have to make the separate assumption, in c), that $(\Phi)^\pm (x' - \bar{c})^\pm \cdot \bar{m}^\pm \in L^1(\rho_t^\pm)$. Note however that if K9 in Proposition 2.3 is fulfilled then, from (2.37), it follows that on the crack-front $(x' - \bar{c})^\pm = (\dot{x} + \underline{F} \bar{\tilde{c}} - \bar{c})^\pm = \dot{x}^\pm + \underline{F}^\pm \bar{\tilde{c}} - \bar{c}$ is parallel to $\bar{e}$ and then (on the crack-front) $(x' - \bar{c})^\pm \cdot \bar{m}^\pm = 0$.

Consider a moving circular tube $\mathcal{T}_{u_t}(R)$, $t \in T$ accompanying the crack-front in the referential description (the definition of a moving circular tube is given in (2.70)). Its image under a family of fracture-deformations $X(\cdot, t)$, $t \in T$ is given by the moving surface

$$\mathcal{T}_{us_t}(R) \equiv X(\mathcal{T}_{u_t}(R), t), \quad t \in T. \quad (2.128)$$

Let

$$\bar{v} = \bar{v}_t(x, R), \quad x \in \mathcal{T}_{us_t}(R) \quad (2.129)$$

denote the normal velocity of $Tu_t(R)$.

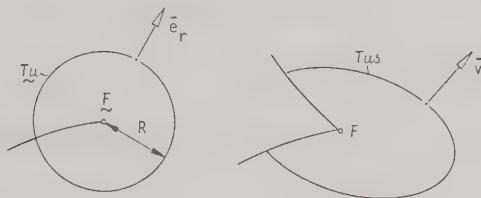


Figure 2.19

Lemma 2.11

$$\bar{v} = \bar{N}(\bar{N} \cdot x'), \quad x \in Tu_t(R) \quad (2.130)$$

where $\bar{N} = \bar{N}_t(x, R)$ is the outward unit normal on $Tu_t(R)$ (see Figure 2.19 above).

Proof: From (2.70) and (2.123) we have

$$Tu_t(R) = \{x \in E \mid x = \hat{X}(s, \theta, t) \equiv \chi(\hat{X}(s, t) + \bar{e}_r(s, \theta, t) R, t)\}$$

and then (at $x \in Tu_t(R)$)

$$\bar{v} = \bar{N}(\bar{N} \cdot \frac{\partial}{\partial t} \hat{X}) = \bar{N}(\bar{N} \cdot (\bar{F}(\frac{\partial}{\partial t} \hat{X} + R \frac{\partial}{\partial t} \bar{e}_r) + \dot{x})) = \bar{N}(\bar{N} \cdot x') +$$

$$\bar{N}((\bar{e} \cdot \frac{\partial}{\partial t} \hat{X}) \bar{N} \cdot \bar{F} \bar{e} + R \bar{N} \cdot \bar{F} \frac{\partial}{\partial t} \bar{e}_r)$$

But

$$\bar{N} \cdot \bar{F} \bar{e} = \bar{F}^T \bar{N} \cdot \bar{e} = A_{Tu}^{-1} J \bar{e}_r \cdot \bar{e} = 0$$

$$\bar{N} \cdot \bar{F} \frac{\partial}{\partial t} \bar{e}_r = \bar{F}^T \bar{N} \cdot \frac{\partial}{\partial t} \bar{e}_r = A_{Tu}^{-1} J \bar{e}_r \cdot \frac{\partial}{\partial t} \bar{e}_r = 0$$

where we have used the geometric relation $\bar{F}^T \bar{N} = A_{Tu}^{-1} J \bar{e}_r$ (A_{Tu} is the area-ratio corresponding to the restriction of $\chi(\cdot, t)$ to $Tu_t(R)$), and this proves the Lemma. $\square$

Remark: The content of Lemma 2.11 was first discussed in Strifors [1]. That paper also provides a heuristic motivation for the boundedness of x' based on the kinematical assumption that, for any $r_0 > 0$ there is a $R_0 > 0$ such that if $0 < R < R_0$ then $Tu_{\mathcal{S}_t}(R) \subset TN_t(r_0)$ for all $t \in T$.

We close this section by presenting some simple relations involving higher order derivatives.

Lemma 2.12 Let $X(\cdot, t)$, $t \in T$ be a family of fracture-deformations satisfying (along with K1 - K8)

K12: The field $\dot{\tilde{\mathcal{L}}}_t$, defined by

$$\dot{\tilde{\mathcal{L}}}_t(X) \equiv \lim_{q \rightarrow 0} \frac{1}{q} (\tilde{\mathcal{L}}_{t+q}(X) - \tilde{\mathcal{L}}_t(X)), \quad X \in TN_t(r_0) \setminus \mathcal{S}_t \quad (2.131)$$

exists.

and let Φ be a C^2 -fracture-field. Then we have

$$\ddot{\Phi} = \Phi'' - 2(\text{Grad} \Phi') \tilde{\mathcal{L}} - (\text{Grad} \Phi) \dot{\tilde{\mathcal{L}}} + (\text{Grad}^2 \Phi)(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}) \quad (2.132)$$

$X \in TN_t(r_0) \setminus \mathcal{S}_t$.

($\text{Grad}^2 \Phi \equiv \text{Grad}(\text{Grad} \Phi)$, $\text{Grad}^2 \Phi \in L_S^2(V, \mathbb{R})$ if Φ is the scalar-valued and $\text{Grad}^2 \Phi \in L_S^2(V, V)$ if Φ is vector-valued).¹

Proof: By reiterated use of formula (2.74) we obtain

$$\begin{aligned} \ddot{\Phi} &= (\dot{\Phi})' - (\text{Grad} \dot{\Phi}) \tilde{\mathcal{L}} = (\Phi' - (\text{Grad} \Phi) \tilde{\mathcal{L}})' - (\text{Grad}(\Phi' - (\text{Grad} \Phi) \tilde{\mathcal{L}})) \tilde{\mathcal{L}} = \\ &= \Phi'' - 2(\text{Grad} \Phi') \tilde{\mathcal{L}} - (\text{Grad} \Phi) \tilde{\mathcal{L}}' + (\text{Grad}((\text{Grad} \Phi) \tilde{\mathcal{L}})) \tilde{\mathcal{L}} = \Phi'' - 2(\text{Grad} \Phi') \tilde{\mathcal{L}} - \\ &= (\text{Grad} \Phi) \dot{\tilde{\mathcal{L}}} + (\text{Grad}^2 \Phi)(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}) \end{aligned}$$

where we have used the relations $(\text{Grad} \tilde{\mathcal{L}}) \tilde{\mathcal{L}} = 0$, $\tilde{\mathcal{L}}' = \dot{\tilde{\mathcal{L}}}$. $\square$

Remark 1: A simple calculation gives

$$\begin{aligned} \text{Grad} \tilde{\mathcal{L}} &= \frac{1}{1 - r \kappa \cos \theta} \left(\frac{\partial}{\partial S} \tilde{\mathcal{L}} \right) \otimes \tilde{\mathcal{L}} = \\ &= \frac{1}{1 - r \kappa \cos \theta} \left((\tilde{\mathcal{E}} \kappa_{\mathcal{G}} + \tilde{\mathcal{M}} \tau_{\mathcal{G}}) \tilde{\mathcal{L}} + \tilde{\mathcal{V}} \frac{\partial}{\partial S} \tilde{\mathcal{L}} \right) \otimes \tilde{\mathcal{L}} \end{aligned} \quad (2.133)$$

¹ $L_S^2(V, \mathbb{R})$ = the space of symmetric bilinear mappings $V \times V \rightarrow \mathbb{R}$.

where $\kappa_g = \kappa_{g,t}(s) = \kappa_t(s) \sin \varphi_t(s)$ is the geodesic curvature of $\tilde{F}_t$ on $\tilde{S}_t$ and $\tau_g = \tau_{g,t}(s) \equiv \tau_t(s) + (d\varphi_t/ds)(s)$ is the geodesic torsion (the angle $\varphi_t = \varphi_t(s)$ is defined in Figure 2.16 a).

Remark 2: Formula (2.132) may also be written

$$\ddot{\Phi} = \Phi'' - 2 \underline{\zeta} (\text{grad } \Phi') \bar{\nu} - (\text{grad } \Phi) \bar{d} + \underline{\zeta} (\text{grad } \Phi) (\text{grad}(\bar{\nu} \underline{\zeta})) \bar{\nu} + \underline{\zeta}^2 (\text{grad}^2 \Phi) (\bar{\nu}, \bar{\nu}) \quad (2.134)$$

where $\bar{d} \equiv \underline{F} \dot{\underline{\zeta}}$.

Corollary 2.4

$$\ddot{X} = X'' - 2 \underline{\zeta} (\text{Grad } X') \bar{\nu} - \underline{F} \dot{\underline{\zeta}} + \underline{\zeta}^2 {}^2F(\bar{\nu}, \bar{\nu}) \quad (2.135)$$

$X \in \tilde{N}_t(r_0) \setminus \tilde{S}_t$, where ${}^2F \equiv \text{Grad}^2 X$.

Proof: Apply Lemma 2.12 with $\Phi = X$. $\square$

Remark: Note that $\text{Grad } X' = \underline{F}' + \underline{F} \text{Grad } \bar{\zeta}$ and thus $(\text{Grad } X') \bar{\nu} = \underline{F}' \bar{\nu}$.

Let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations (satisfying K1 - K8 and K12). We say that steady state conditions for X are prevailing at the crack-front if

$$\begin{aligned} X'(X, t) &= \bar{0} \\ \dot{\underline{\zeta}}_t(X) &= \bar{0} \end{aligned} \quad X \in \tilde{N}_t(r_0) \setminus \tilde{S}_t, \quad t \in T \quad (2.136)$$

Proposition 2.13 At steady state we have

$$\dot{X} = - \underline{\zeta} \underline{F} \bar{\nu}, \quad \ddot{X} = \underline{\zeta}^2 {}^2F(\bar{\nu}, \bar{\nu}) \quad (2.137)$$

Proof: This is a trivial consequence of (2.76), (2.135) and (2.136). $\square$

Remark 1: In a non steady state situation it may be justifiable to assume that the fields X' and X'' are bounded at the crack-front and that ${}^2F(\bar{\nu}, \bar{\nu})$ has a "stronger" singularity than $\underline{F} \dot{\underline{\zeta}}$ and $(\text{Grad } X') \bar{\nu}$. In that case we have asymptotically (as $\underline{F}$ is approached from $B \setminus \underline{S}$)

$$\ddot{X} \approx \underline{\zeta}^2 {}^2F(\bar{\nu}, \bar{\nu}) \quad (2.138)$$

Remark 2: The results contained in formulas (2.135) and (2.137) are previously discussed in the literature. See for instance Atkinson & Eshelby [10], Sih [11], Strifors [1, 2] and Gurtin & Yatomi [9].

2.4 Singular surfaces

Let $\chi(\cdot, t)$, $t \in T$ be a family of fracture-deformations. According to the assumptions made in section 2.2 the fields $\dot{\chi}(\cdot, t)$, $\underline{F}(\cdot, t)$, $\ddot{\chi}(\cdot, t)$, $\dot{\underline{F}}(\cdot, t)$, ... are continuous on $B \setminus \underline{\Sigma}_t$. In other words, the region $B \setminus \underline{\Sigma}_t$ is assumed to be free of shock waves, acceleration waves and higher order singular surfaces. In this section we will supplement our kinematical description of a fracturing body with the possibility that, in addition to the crack-surface, we may have singular surfaces for χ of order greater than or equal to one. (Away from the crack-front the crack-surface may be considered as a singular surface of order zero).

Singular surfaces are extensively discussed in Truesdell & Toupin [5].¹ Here we just present some basic notation and a few elementary results required for the discussion of the balance laws.

Let $\underline{\Sigma}_t$, $t \in T$ be a smooth moving surface in E and let $\bar{\underline{M}}_t$ and $\bar{\underline{w}}_t$ denote its orientation and normal-velocity respectively. The intersection with the regular region $B \setminus (\underline{\Sigma}_t \cap B)$ will again be denoted $\underline{\Sigma}_t$. The image of $\underline{\Sigma}_t$ under χ is given by the moving surface

$$\Sigma_t \equiv \chi(\underline{\Sigma}_t, t), \quad t \in T \quad (2.139)$$

with orientation $\bar{\underline{M}}_t$ and normal-velocity $\bar{\underline{w}}_t$. We may write $\bar{\underline{w}} = \bar{\underline{M}} \underline{w}$ and $\bar{\underline{w}} = \bar{\underline{M}} \underline{w}$.

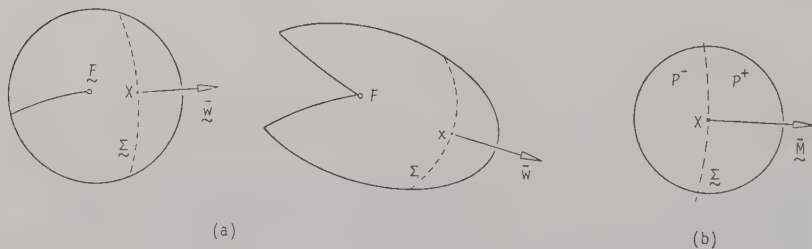


Figure 2.20

¹ sect. 172 - 194.

Let Φ be a fracture-field. The jump in $\Phi(\cdot, t)$ across Σ_t is denoted $[\Phi](\cdot, t)$ and defined by

$$[\Phi](X, t) \equiv (\Phi)^+(X, t) - (\Phi)^-(X, t), \quad X \in \Sigma_t \setminus \underline{\Sigma}_t \quad (2.140)$$

(The symbols $(\Phi)^+$, $(\Phi)^-$ are again adopted to denote one-sided limits as $X \in \Sigma_t \setminus \underline{\Sigma}_t$ is approached from the "positive" and "negative" side respectively. Cf. the construction in connection with definition (2.1). We assume that Σ_t divides a regular subregion P into two regular pieces $P^+(t)$ and $P^-(t)$. See Figure 2.20(b).)

Lemma 2.13 Let Σ_t , $t \in T$ be a singular surface for X of order ≥ 1 . Assume that

K13: for $X \in \Sigma_t \setminus \underline{\Sigma}_t$ there is a subregion $P \ni X$, an open time-interval $T^- \ni t$ and C^2 -mappings

$$P \times T^- \ni (Y, s) \sim \chi_e^\pm(Y, s) \in E \quad (2.141)$$

satisfying: $\chi(Y, s) = \chi_e^+(Y, s)$ for $Y \in P^+(s)$ and $\chi(Y, s) = \chi_e^-(Y, s)$ for $Y \in P^-(s)$, $s \in T^-$.

Then

$$w_t(x) - (\dot{\chi})^\pm(x, t) \cdot \bar{M}_t(x) = w_t(X)(A_{\Sigma, t}(X))^{-1}(J)^\pm(X, t) \quad (2.142)$$

$x = \chi(X, t)$, $X \in \Sigma_t \setminus \underline{\Sigma}_t$, where $(\dot{\chi})^\pm$, $(J)^\pm$ are limits in the sense discussed above and $A_{\Sigma, t}$ is the area-ratio associated with the restriction of $\chi(\cdot, t)$ to Σ_t .

Proof: Let $\hat{X}$ be a parametrization of the singular surface in the referential description, i.e.

$$\Sigma_t = \{X \in E \mid X = \hat{X}(q_1, q_2, t), (q_1, q_2) \in U\}, \quad t \in T$$

(U is an open subset of $\mathbb{R}^2$) then, in the spatial description, we have

$$\Sigma_t = \{x \in E \mid x = \hat{\chi}(q_1, q_2, t), (q_1, q_2) \in U\}, \quad t \in T$$

where

$$\hat{x}(q_1, q_2, t) = \chi(\hat{X}(q_1, q_2, t), t) = \chi_e^\pm(\hat{X}(q_1, q_2, t), t)$$

Then, at $X \in \Sigma_t \setminus S_t$, we have (with $F_e^\pm \equiv \frac{\partial}{\partial V} \chi_e^\pm$ and $\dot{x}_e^\pm \equiv \frac{\partial}{\partial t} \chi_e^\pm$)

$$w = \bar{M} \cdot \left(\frac{\partial}{\partial t} \hat{x} \right) = \bar{M} \cdot \left(F_e^\pm \frac{\partial}{\partial t} \hat{X} + \dot{x}_e^\pm \right) = \bar{M} \cdot \left((F)^\pm \frac{\partial}{\partial t} \hat{X} + (\dot{x})^\pm \right) \quad (2.143)$$

But

$$\frac{\partial}{\partial t} \hat{X} = \bar{M} \underline{w} + \bar{w}_{||}, \quad \bar{M} \cdot \bar{w}_{||} = 0 \quad (2.144)$$

and from (2.143), (2.144) we get

$$w - (\dot{x})^\pm \cdot \bar{M} = \bar{M} \cdot (F)^\pm (\bar{M} \underline{w} + \bar{w}_{||}) \quad (2.145)$$

and formula (2.142) now follows from (2.145) and the geometric relation

$$\bar{M} = A_\Sigma^{-1} (J)^\pm ((F)^\pm)^{-T} \bar{M}$$

□

Remark 1: The scalar fields

$$w^\pm = w_t^\pm(x) \equiv w_t(x) - (\dot{x})^\pm(x, t) \cdot \bar{M}_t(x), \quad x \in \Sigma_t \quad (2.146)$$

are called intrinsic wave-front speeds. From Lemma 2.13 it follows that

$$[w] \equiv w^+ - w^- = \underline{w} A_\Sigma [J] \quad (2.147)$$

and thus if $[J] = 0$ then $[w] = 0$ (cf. Truesdell & Toupin [5], section 189).

Remark 2: With notation as in the proof of Lemma 2.13 we have the relation

$$(\dot{x})^\pm = - \underline{w} (F)^\pm \bar{M} - (F)^\pm \bar{w}_{||} + \frac{\partial}{\partial t} \hat{x}$$

and since $[F] \bar{w}_{||} = \bar{0}$ we obtain the important relation

$$[\dot{x}] = - \underline{w} [F] \bar{M} \quad (2.148)$$

Proposition 2.14 Let P be a "non-cracking" subbody of B , i.e. a regular subregion of B satisfying $\sum_t \cap \bar{P} = \emptyset$ for all $t \in T$, and let ϕ be a C^1 -fracture-density. Assume that $\sum_t$, $t \in T$ is a singular surface for X of order ≥ 1 , satisfying K13, and a singular surface for ϕ of order ≥ 0 . Then

$$\frac{d}{dt} \int_{P_t} \phi dv(x) = \int_{P_t} (\dot{\phi} + \phi \operatorname{div} \dot{X}) dv(x) - \int_{\sum_t \cap P_t} [\phi \omega] da(x) \quad (2.149)$$

Proof: We have (see Figure 2.20 (b))

$$\begin{aligned} \frac{d}{dt} \int_{P_t} \phi dv(x) &= \frac{d}{dt} \int_P \phi_p dv(X) = \frac{d}{dt} \int_{P^+(t)} \phi_p dv(X) + \frac{d}{dt} \int_{P^-(t)} \phi_p dv(X) = \\ &= \int_{P^+(t)} \dot{\phi}_p dv(X) - \int_{\sum_t \cap P} (\phi_p)^+ \omega da(X) + \int_{P^-(t)} \dot{\phi}_p dv(X) + \int_{\sum_t \cap P} (\phi_p)^- \omega da(X) = \\ &= \int_P \dot{\phi}_p dv(X) - \int_{\sum_t \cap P} [\phi_p] \omega da(X) \end{aligned} \quad (2.150)$$

where we have used the transport theorem in Appendix A.9. From Lemma 2.13 we obtain the relation

$$[\phi_p] \omega = [\phi \omega] A_\Sigma, \quad X \in \sum_t \cap P \quad (2.151)$$

and formula (2.149) now follows from (2.150), (2.151) and (2.46)₁. $\square$

3. EQUATIONS OF BALANCE

Continuum mechanics is based on equations expressing the balance of mass, momentum, moment of momentum and energy for a body, and an inequality guaranteeing the non-negative production of entropy. These basic physical laws apply to all material bodies, irrespective of their constitution (solid, fluid, elastic, visco-elastic, etc) and in all their motions (regular or involving waves or cracks).

The mathematical representation of these physical principles are all formally alike. They have the general form:

$$\frac{d}{dt}B(P,t) = P(P,t) + S(P,t), \quad t \in T$$

for all subbodies P of B . In this equation $B(\cdot,t)$, $P(\cdot,t)$ and $S(\cdot,t)$ are time-dependent scalar- or vector-valued measures on the set of all subbodies of B . Here B is called the balanced quantity (momentum, energy, etc), P is called the production of the balanced quantity within P (is equal to zero except in the case of entropy) and S the supply of the balanced quantity from sources outside P (force, working, heating, etc). Various continuum theories now deal with different representations for B , P and S in terms of appropriate fields associated with the body and its motion. Absolute continuity of B , P and S with respect to mass-, volume- or area-measures is a prescription usually employed, leading up to integral statements of the balance laws (non-local theories are not considered in this paper).

It is the purpose of this chapter to postulate the equations of balance for a fracturing body. We introduce the fields used for the representation of B , P and S and by utilizing their smoothness properties we derive local representatives for our global balance laws. The admittance of fields that are essentially singular (unbounded) at the crack-front will create a number of non-trivial problems.

The fields and the equations are discussed in an abstract setting, leaving open, for the moment, their concrete thermo-mechanical interpretation. In this way the structural characteristics of our equations are clearly displayed. A specific thermo-mechanical model is presented in chapter 5.

3.1 Master balance law

Throughout this chapter we assume that $X(\cdot, t)$, $t \in T$ is a family of fracture-deformations satisfying K1 - K8, K9 (or K9', K9''), K10 and K13. A subbody P of B is identified by a fixed, regular subregion P of B . At any given fixed time $t \in T$ we consider the following three disjunct classes of subbodies:

B1: The class of all subbodies P of B for which $\bar{P} \cap \underline{S}_t = \emptyset$. Members of this class are called non-cracking subbodies at time t .

If P is a non-cracking subbody at time t then there is an open time-interval $T' \ni t$ such that P is non-cracking for all times in T' (i.e. $\bar{P} \cap \underline{S}_s = \emptyset$ for all $s \in T'$).

B2: The class of all subbodies P of B for which $\bar{P} \cap \underline{S}_t \neq \emptyset$ and $\bar{P} \cap \underline{F}_t = \emptyset$ (i.e. $\bar{P} \cap \underline{S}_t \subset \underline{S}_t$). Members of this class are called cracked subbodies at time t .

The cracked subbody P is thus divided into two parts P^+ and P^- by the surface $\underline{S}_t$. We agree to consider P^+ and P^- as separate subbodies of B belonging to the class B2. Note that if P is a cracked subbody at time t then there is an open time-interval $T' \ni t$ such that P is cracked for all times in T' .

B3: The class of all subbodies P of B for which $\bar{P} \cap \underline{S}_t \neq \emptyset$, $\bar{P} \cap \underline{F}_t \neq \emptyset$ and $\underline{S}_s \cap \bar{P}$, $s \in T'$ is a moving, regular subregion of $\underline{S}_s$ for some open time-interval $T' \ni t$. Members of this class are called cracking subbodies at time t .

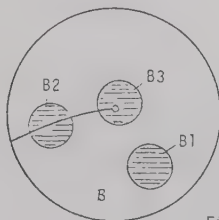


Figure 3.1

A subbody may, of course, change class in the course of time as the crack-front is progressing. Note that in chapter 2 we have assumed that our primary body B belongs to class B3 for all times $t \in T$.

We introduce the following fracture-fields:

Φ	volumedistributed <u>balanced quantity</u>
Π	volumedistributed <u>production</u> of Φ
Q	volumedistributed <u>supply</u> of Φ
K	volumedistributed <u>efflux</u> of Φ

Here Φ , Π and Q are tensor-fields of the same order ($\Phi(X,t) \in \mathbb{R}$ or V) and K is a tensor-field of order greater by one than that of Φ ($K(X,t) \in V$ or $\text{End}(V)$). We assume that

1°. Φ is a C^1 -fracture-density and that for every $P \in B_3$ at time t there is an open time-interval $T^- \ni t$ such that the function

$$T^- \ni s \leadsto \int_{P_S} \Phi(x,s) dv(x) \quad (3.1)$$

is differentiable.

2°. Π and Q are C^0 -fracture-densities.

3°. K is a C^1 -fracture-field and, with P and T^- as in 1°, we have $K\bar{n} \in L^1(\partial P_S \setminus (p_S^+ \cup p_S^-))$ for all $s \in T^-$ ($p_t^\pm \equiv \chi^\pm(S_t \cap P, t)$).

Remark: Note that if P is a non-cracking or cracked subbody at time t then it trivially follows that (for some time-interval $T^- \ni t$) the function (3.1) is differentiable and $K\bar{n} \in L^1(\partial P_S \setminus (p_S^+ \cup p_S^-))$ for every $s \in T^-$. The derivative of (3.1) is given by the formula (2.50).

Next we introduce the following crack-surface-fields:

$\Psi^\pm$	surfacedistributed <u>balanced quantity</u>
$\Pi_S^\pm$	surfacedistributed <u>production</u> of $\Psi^\pm$
$R^\pm$	surfacedistributed <u>supply</u> of $\Psi^\pm$
$k^\pm$	surfacedistributed <u>efflux</u> of $\Psi^\pm$

Here Ψ , Π_S and R are tensor-fields of the same order as Φ ($\Psi^\pm(X,t) \in \mathbb{R}$ or V) and k is a tensor-field of order greater by one than that of Ψ ($k^\pm(X,t) \in T_{X^\pm}^\pm(X,t)$, $\text{End}(T_{X^\pm}^\pm(X,t))$ or $L(T_{X^\pm}^\pm(X,t), V)$).

We assume that

4°. $\psi^\pm$ are C^1 -crack-surface-fields satisfying F6 (or F6'), (see Proposition 2.8 and 2.8').

5°. $\pi_S^\pm$ and $R^\pm$ are C^0 -crack-surface-fields.

6°. $k^\pm$ are C^1 -crack-surface-fields satisfying $\operatorname{div}_\delta k^\pm(\cdot, t) \in L^p(S_t^\pm)$, $p > 1$. (Appendix A.8)

Finally we introduce the crack-front-field:

π_F - curvedistributed production of the balanced quantity

a tensor-field defined on $\tilde{F}$ and of the same order as Φ , and the singular-surface-field:

π_Σ - surfacedistributed production of the balanced quantity

a tensor-field defined on $\Sigma \setminus \tilde{F}$ and of the same order as Φ . We assume that

7°. $\pi_F(\cdot, t)$ and $\pi_\Sigma(\cdot, t)$ are continuous functions on $\tilde{F}_t$ and $\Sigma_t \setminus \tilde{F}_t$ respectively and $\pi_\Sigma(\cdot, t) \in L^1(\Sigma_t)$.

Remark: Crack-propagation and singular-surface-propagation in materials are, in most cases, irreversible processes and one expects them to be accompanied with irreversible entropy production.

We say that P is an interior subbody of B if $P \subset \overset{\circ}{B}$, otherwise it is called a subbody at the boundary of B .

For an arbitrary interior subbody P , belonging to any of the classes B1, B2 or B3 at time t , we put

$$B(P, t) = \int_{P_t} \Phi(x, t) dv(x) + \int_{P_t^+} \psi^+(x, t) dv(x) + \int_{P_t^-} \psi^-(x, t) dv(x) \quad (3.2)$$

$$P(P, t) = \int_{P_t} \pi(x, t) dv(x) + \int_{P_t^+} \pi_S^+(x, t) da(x) + \int_{P_t^-} \pi_S^-(x, t) da(x) + \\ + \int_{F_t \cap \bar{P}_t} \pi_F(x, t) ds(x) + \int_{\Sigma_t \cap P_t} \pi_\Sigma(x, t) da(x) \quad (3.3)$$

$$\begin{aligned}
S(P, t) \equiv & \int_{P_t} Q(x, t) dv(x) + \int_{\partial P_t \setminus (p_t^+ \cup p_t^-)} K(x, t) \bar{n}_t(x) da(x) + \\
& \int_{p_t^+} R^+(x, t) da(x) + \int_{\partial p_t^+ \setminus (F_t \cap \bar{p}_t^+)} k^+(x, t) \bar{v}_t^+(x) ds(x) + \\
& \int_{p_t^-} R^-(x, t) da(x) + \int_{\partial p_t^- \setminus (F_t \cap \bar{p}_t^-)} k^-(x, t) \bar{v}_t^-(x) ds(x) \quad (3.4)
\end{aligned}$$

where $p_t^\pm \equiv \chi^\pm(\Sigma_t \cap P, t)$, $\bar{n} \equiv \bar{n}_t(x)$ is the outward pointing unit normal on ∂P_t and $\bar{v}^\pm = \bar{v}_t^\pm(x)$ is the outward pointing unit normal on $\partial p_t^\pm$. See the illustration below!

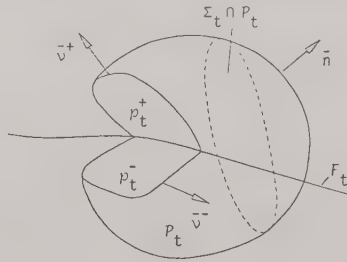


Figure 3.2

Note that for a non-cracking subbody we have $p_t^\pm = \emptyset$ and for a cracked subbody $F_t \cap \bar{p}_t = \emptyset$. We state the

Master balance law: With B , P and S defined by (3.1) - (3.4) we have

$$\frac{d}{dt} B(P, t) - P(P, t) - S(P, t) = 0 \quad (\text{Mb1})$$

for all interior subbodies P belonging to $B1$, $B2$ or $B3$ at time t .

This balance law is the basic and general formulation of our physical principles. It is postulated as an axiom of the theory. Unfortunately, the integral equation (Mb1) is very cumbersome as it stands and it has to be replaced, if possible, by local representatives (differential equations) in order to make the theory workable.

In what follows we derive localized balance laws associated with the following generic material points of B :

regular bulk material point X

singular bulk-material point	x^i
crack-surface material point	x^{ii}
crack-front material point	x^{iii}

and at the end of this section we discuss the local balance law at a

boundary material point	x^{iv}
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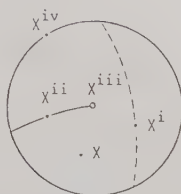


Figure 3.3

Proposition 3.1 (Local balance law 1)

$$\dot{\Phi} + \Phi \operatorname{div} \dot{x} - \Pi - Q - \operatorname{div} K = 0 \quad (\text{Lb11})$$

$\mathcal{B}_t \ni x = X(X, t)$ and X is a regular bulk-material point.

Proof: Let P be a non-cracking interior subbody at time t . From (Mb1) we have (no singular surface)

$$\frac{d}{dt} \int_{P_t} \Phi \, dv(x) - \int_{P_t} \Pi \, dv(x) - \int_{P_t} Q \, dv(x) - \int_{\partial P_t} K \bar{n} \, da(x) = 0 \quad (3.5)$$

The regularity of X , Φ and K on P permit us to use formula (cf. (2.50))

$$\frac{d}{dt} \int_{P_t} \Phi \, dv(x) = \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x}) \, dv(x) \quad (3.6)$$

and the divergence theorem

$$\int_{\partial P_t} K \bar{n} \, da(x) = \int_{P_t} \operatorname{div} K \, dv(x) \quad (3.7)$$

(3.5) - (3.7) give

$$\int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \Pi - Q - \operatorname{div} K) dv(x) = 0 \quad (3.8)$$

Let $\bar{B}_t \ni x = X(X, t)$ where X is a regular bulk-material point. By a standard argument, involving (3.8), the arbitrariness of the selected non-cracking interior subbody P and the smoothness of the fracture-fields on P , we conclude that

$$(\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \Pi - Q - \operatorname{div} K)(x, t) = 0$$

□

Remark 1: The local balance law 1 may also be written

$$\partial_t \Phi - \Pi - Q - \operatorname{div}(K - \Phi \dot{X}) = 0 \quad (3.9)$$

Remark 2: Note that from (Lb11) and 2^0 it follows that

$$\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \operatorname{div} K = \Pi + Q \in L^1(\bar{B}_t)$$

Proposition 3.2 (Local balance law 2)

$$[\Phi \omega + K \bar{M}] + \Pi_\Sigma = 0 \quad (\text{Lb12})$$

$\bar{B}_t \ni x = X(X, t)$ and X is a singular bulk-material point (ω and $\bar{M}$ are introduced in section 2.4).

Proof: Let P be a non-cracking interior subbody at time t containing the singular surface $\Sigma_t \cap P$. From (Mb1) we have

$$\frac{d}{dt} \int_{P_t} \Phi dv(x) - \int_{P_t} \Pi dv(x) - \int_{\Sigma_t \cap P_t} \Pi_\Sigma da(x) - \int_{P_t} Q dv(x) - \int_{\partial P_t} K \bar{n} da(x) = 0 \quad (3.10)$$

We may now use formula (2.149) and the divergence theorem

$$\int_{\partial P_t} K \bar{n} da(x) = \int_{P_t} \operatorname{div} K dv(x) + \int_{\Sigma_t \cap P_t} [K] \bar{M} da(x)$$

to obtain

$$\int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \Pi - Q - \operatorname{div} K) dv(x) - \int_{\Sigma_t \cap P_t} ([\Phi \omega + K \bar{M}] + \Pi_\Sigma) da(x) = 0 \quad (3.11)$$

where the first integral is equal to zero according to (Lb11). Thus

$$\int_{\Sigma_t \cap P_t} ([\Phi \omega + K \bar{M}] + \pi_{\Sigma}') da(x) = 0 \quad (3.12)$$

Let $\tilde{B}_t^0 \ni x = \chi(X, t)$ where X is a singular bulk-material point. By a standard argument, involving (3.12), the arbitrariness of the selected non-cracking interior subbody P and the continuity of $([\Phi \omega + K \bar{M}] + \pi_{\Sigma})(\cdot, t)$ on $\Sigma_t \setminus F_t$, we conclude that

$$([\Phi \omega + K \bar{M}] + \pi_{\Sigma})(x, t) = 0$$

□

Remark: Propositions 3.1 and 3.2 are classical results discussed for instance in Truesdell & Toupin [5], sections 157 and 193.

Proposition 3.3 (Local balance law 3)

$$\dot{\Psi}^{\pm} + \Psi^{\pm} \operatorname{div}_{\delta} \dot{x}^{\pm} - \pi_S^{\pm} - R^{\pm} + (K)^{\pm} \bar{m}^{\pm} - \operatorname{div}_{\delta} k^{\pm} = 0 \quad (\text{Lb1.3})$$

$\tilde{S}_t^{\pm} \ni x = \chi^{\pm}(X, t)$, $X \in \tilde{S}_t^{\pm}$ (a crack-surface material point).

Proof: Let P be a cracked interior subbody at time t . Consider the part P_t^+ . The other part is analogously handled.

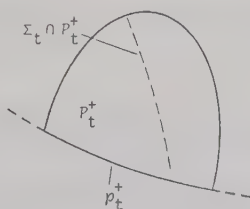


Figure 3.4

From (Mb1) we have

$$\frac{d}{dt} \left(\int_{P_t^+} \Phi dv(x) + \int_{P_t^+} \Psi^+ da(x) \right) - \int_{P_t^+} \pi dv(x) - \int_{P_t^+} \pi_S^+ da(x) - \int_{\Sigma_t \cap P_t^+} \pi_{\Sigma}^+ da(x) -$$

$$\int_{P_t^+} Q dv(x) - \int_{\partial P_t^+ \setminus P_t^+} K \bar{n} da(x) - \int_{P_t^+} R^+ da(x) - \int_{\partial P_t^+} k^+ \bar{v}^+ ds(x) = 0 \quad (3.13)$$

The regularity of X , Φ and K on P^+ (except for the jump-discontinuity on $\Sigma_t \cap P$) and the regularity of X^+ , Ψ^+ and k^+ on $\Sigma_t \cap P$ allow us to use formula (2.149), the derivative

$$\frac{d}{dt} \int_{P_t^+} \Psi^+ da(x) = \int_{P_t^+} (\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\delta} \dot{X}^+) da(x)$$

(cf. formula (2.58) with $c^+ = 0$) and the divergence theorems

$$\int_{\partial P_t^+ \setminus P_t^+} K \bar{n} da(x) = \int_{P_t^+} \operatorname{div} K dv(x) + \int_{\Sigma_t \cap P_t^+} [K] \bar{M} da(x) - \int_{P_t^+} (K)^+ \bar{m}^+ da(x)$$

and

$$\int_{\partial P_t^+} k^+ \bar{v}^+ ds(x) = \int_{P_t^+} \operatorname{div}_{\delta} k^+ da(x)$$

to show that (3.13) is equivalent to

$$\int_{P_t^+} (\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \Pi_- - Q - \operatorname{div} K) dv(x) - \int_{\Sigma_t \cap P_t^+} ([\Phi \omega + K \bar{M}] + \Pi_{\Sigma}) da(x) +$$

$$\int_{P_t^+} (\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\delta} \dot{X}^+ - \Pi_S^+ - R^+ + (K)^+ \bar{m}^+ - \operatorname{div}_{\delta} k^+) da(x) = 0$$

where the two first integrals are equal to zero according to (Lb11) and (Lb12). Thus

$$\int_{P_t^+} (\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\delta} \dot{X}^+ - \Pi_S^+ - R^+ + (K)^+ \bar{m}^+ - \operatorname{div}_{\delta} k^+) da(x) = 0 \quad (3.14)$$

Let $\tilde{S}_t^+ \ni x = X^+(X, t)$, $X \in \tilde{\Sigma}_t^+$. By a standard argument, involving (3.14), the arbitrariness of the selected cracked subbody P and the continuity of $(\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\delta} \dot{X}^+ - \Pi_S^+ - R^+ + (K)^+ \bar{m}^+ - \operatorname{div}_{\delta} k^+)(\cdot, t)$ on $S_t^+ \setminus F_t$, we conclude that

$$(\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\delta} \dot{X}^+ - \Pi_S^+ - R^+ + (K)^+ \bar{m}^+ - \operatorname{div}_{\delta} k^+)(x, t) = 0$$

□

Remark 1: We have here assumed that $k^\pm(x, t) \in T_x^\pm$, $\text{End}(T_x^\pm)$ or $L(T_x^\pm, V)$. If $k^\pm(x, t) \in V$ or $\text{End}(V)$ then (Lb13) is changed for

$$\ddot{\Psi}^\pm + \Psi^\pm \text{div}_\delta \dot{X}^\pm - \Pi_S^\pm \cdot R^\pm + (K)^\pm \bar{m}^\pm - \text{div}_\delta k^\pm - 2M^\pm k^\pm \bar{m}^\pm = 0$$

where $M^\pm$ are the mean curvatures defined in (2.24) (cf. Appendix A.8).

Remark 2: The local balance law 3 may also be written

$$\partial_t \Psi^\pm - 2M^\pm \dot{X}^\pm \cdot \bar{m}^\pm \Psi^\pm - \Pi_S^\pm \cdot R^\pm + (K)^\pm \bar{m}^\pm - \text{div}_\delta (k^\pm - \Psi^\pm \otimes \dot{X}_\parallel^\pm) = 0$$

where $\partial_t \Psi^\pm$ is defined by

$$\partial_t \Psi^\pm \equiv \dot{\Psi}^\pm - (\text{grad}_\delta \Psi^\pm) \cdot \dot{X}_\parallel^\pm$$

Note that $\partial_t \Psi^\pm$ may, in general, not be defined as the partial derivative with respect to t of the function $\Psi^\pm(x, t)$, $x \in S_t^\pm$.

We have the simple observation

Corollary 3.1 $(K)^\pm \bar{m}^\pm \in L^1(S_t^\pm)$.

Proof: From (Lb13) we have

$$(K)^\pm \bar{m}^\pm = \Pi_S^\pm + R^\pm + \text{div}_\delta k^\pm - \dot{\Psi}^\pm - \Psi^\pm \text{div}_\delta \dot{X}^\pm, \quad x \in \bar{S}_t^\pm$$

The assumptions made in 4^o - 6^o implicate that $\Pi_S^\pm(\cdot, t)$ and $R^\pm(\cdot, t)$ are continuous on $S_t^\pm$ and that $\text{div}_\delta k^\pm - \dot{\Psi}^\pm - \Psi^\pm \text{div}_\delta \dot{X}^\pm \in L^1(S_t^\pm)$. This proves the Corollary. $\square$

We introduce the following simplifying notation: If $\Psi^\pm$ are C^0 -crack-surface-fields then

$$]\Psi[(\cdot, t) : \tilde{F}_t \rightarrow \mathbb{R}, V, \dots$$

is defined by

$$]\Psi[(X, t) \equiv \Psi^+(X, t) + \Psi^-(X, t), \quad X \in \tilde{F}_t$$

Proposition 3.4 (The reduced master balance law)

$$\begin{aligned} & \frac{d}{dt} \int_{P_t} \Phi dv(x) - \int_{P_t} (\Pi + Q) dv(x) - \int_{\partial P_t} K \bar{n} da(x) - \int_{\Sigma_t \cap P_t} \pi_{\Sigma} da(x) + \\ & \int_{F_t \cap \bar{P}_t} (\Psi c + k \bar{v} [-\pi_F] ds(x) = 0 \end{aligned} \quad (\text{Rmb1})$$

for all interior subbodies P belonging to any of the classes B1, B2 or B3 at time t (the intrinsic crack-front speeds $c^{\pm}$ are defined in (2.41)).

Proof: Let P be a non-cracking or cracked interior subbody at time t . Then (Rmb1) follows from (Lb11), (Lb12) and the fact that $F_t \cap \bar{P}_t = \emptyset$. Let P be a cracking interior subbody at time t . From (Mb1) we have

$$\begin{aligned} & \frac{d}{dt} \left(\int_{P_t} \Phi dv(x) + \int_{P_t} \Psi^+ da(x) + \int_{P_t} \Psi^- da(x) \right) - \int_{P_t} \Pi dv(x) - \int_{P_t} \pi_S^+ da(x) - \\ & \int_{P_t} \pi_S^- da(x) - \int_{F_t \cap \bar{P}_t} \pi_F ds(x) - \int_{\Sigma_t \cap P_t} \pi_{\Sigma} da(x) - \int_{P_t} Q dv(x) - \\ & \int_{\partial P_t} \left((P_t^+ \cup P_t^-) K \bar{n} da(x) - \int_{P_t^+} R^+ da(x) - \int_{\partial P_t} (F_t \cap \bar{P}_t) k^+ \bar{v}^+ ds(x) - \int_{P_t^-} R^- da(x) - \right. \\ & \left. \int_{\partial P_t} (F_t \cap \bar{P}_t) k^- \bar{v}^- ds(x) \right) = 0 \end{aligned} \quad (3.15)$$

From Corollary 3.1 it follows that we may write

$$\int_{\partial P_t} \left((P_t^+ \cup P_t^-) K \bar{n} da(x) \right) = \int_{\partial P_t} K \bar{n} da(x) - \int_{P_t^+} (K)^+ \bar{m}^+ da(x) - \int_{P_t^-} (K)^- \bar{m}^- da(x) \quad (3.16)$$

and from 6° and the divergence theorem we have

$$\int_{\partial P_t} \left((F_t \cap \bar{P}_t) k^{\pm} \bar{v}^{\pm} ds(x) \right) = \int_{P_t} \operatorname{div}_{\Delta} k^{\pm} da(x) - \int_{F_t \cap \bar{P}_t} k^{\pm} \bar{v}^{\pm} ds(x) \quad (3.17)$$

Inserting (3.16), (3.17) and formula (2.58) into (3.15) gives

$$\frac{d}{dt} \int_{P_t} \Phi dv(x) - \int_{P_t} (\Pi + Q) dv(x) - \int_{\partial P_t} K \bar{n} da(x) - \int_{\Sigma_t \cap P_t} \pi_{\Sigma} da(x) +$$

$$\int_{P_t} (\dot{\Psi}^+ + \Psi^+ \operatorname{div}_{\Delta} \dot{X}^+ - \Pi_S^+ - R^+ + (K)^+ \bar{m}^+ - \operatorname{div}_{\Delta} k^+) da(x) + \int_{P_t} (\dot{\Psi}^- + \Psi^- \operatorname{div}_{\Delta} \dot{X}^- - \Pi_S^- - R^- + (K)^- \bar{m}^- - \operatorname{div}_{\Delta} k^-) da(x) + \int_{F_t \cap \bar{P}_t} (J\Psi c + k \bar{v} [-\Pi_F]) ds(x) = 0$$

where the two surface-integrals are equal to zero according to (Lb13). This proves the Proposition. $\square$

So far we have demonstrated that the local balance laws 1 - 3 and the reduced master balance law are, under the assumed smoothness of our participating fields, logical consequences of the master balance law. That the converse is also true is readily established and we may summarize our results in

Theorem 3.1 The master balance law is satisfied if and only if the local balance laws 1 - 3 and the reduced master balance law are satisfied.

The statement in Theorem 3.1 is restricted to interior subbodies. We close this section with a brief discussion of the balance law for a subbody P having some part of its boundary in common with the boundary of B . These subbodies will be called subbodies at the boundary of B .

For the sake of simplicity we assume that ∂B is a smooth surface. The family of mappings

$$\chi^{\partial}(\cdot, t) \equiv \chi(\cdot, t)|_{\partial B \setminus \tilde{S}_t} : \partial B \setminus \tilde{S}_t \rightarrow \partial B_t \setminus (S_t^+ \cup S_t^-), \quad t \in T$$

represents the motion of the boundary ∂B . Put

$$\dot{\chi}^{\partial}(X, t) \equiv \partial_t \chi^{\partial}(X, t), \quad \ddot{\chi}^{\partial}(X, t) \equiv \partial_t^2 \chi^{\partial}(X, t)$$

$$\underline{F}^{\partial}(X, t) \equiv \nabla_{\Delta} \chi^{\partial}(X, t) = \underline{R}^{\partial}(X, t) \underline{U}^{\partial}(X, t), \quad A^{\partial}(X, t) \equiv \det \underline{U}^{\partial}(X, t)$$

where $\underline{U}^{\partial}$ is the stretch tensor, $\underline{R}^{\partial}$ the rotation tensor and A^{∂} the area-ratio associated with the deformation of ∂B . We assume that $A^{\partial} \in L^1(\partial B)$. A family of fields

$$\Psi^{\partial}(\cdot, t) : \partial B \setminus \tilde{S}_t \rightarrow \mathbb{R}, \quad V, \dots$$

is called a C^k -boundary-field if

where $p_t^\partial \equiv \chi^\partial(\partial B \cap \bar{P}, t)$, is differentiable.

2^o. Π^∂ and R^∂ are C^0 -boundary-densities.

3^o. k^∂ is a C^1 -boundary-field.

For an arbitrary subbody P (at the boundary of B), belonging to any of the classes B1, B2 or B3 at time t , we put

$$B(P, t) \equiv \int_{P_t} \Phi(x, t) dv(x) + \int_{p_t^+} \Psi^+(x, t) da(x) + \int_{p_t^-} \Psi^-(x, t) da(x) + \int_{p_t^\partial} \Psi^\partial(x, t) da(x) \quad (3.18)$$

$$P(P, t) \equiv \int_{P_t} \Pi(x, t) dv(x) + \int_{p_t^+} \Pi_S^+(x, t) da(x) + \int_{p_t^-} \Pi_S^-(x, t) da(x) + \int_{p_t^\partial} \Pi^\partial(x, t) da(x) + \int_{F_t \cap \bar{P}_t} \Pi_F(x, t) ds(x) + \int_{\Sigma_t \cap \bar{P}_t} \Pi_\Sigma(x, t) da(x) \quad (3.19)$$

$$S(P, t) \equiv \int_{P_t} Q(x, t) dv(x) + \int_{\partial P_t \setminus (p_t^+ \cup p_t^- \cup p_t^\partial)} K(x, t) \bar{n}_t(x) da(x) + \int_{p_t^+} R^+(x, t) da(x) + \int_{\partial p_t^+ \setminus (\partial S_t^+ \cap \bar{P}_t)} k^+(x, t) \bar{v}_t^+(x) ds(x) + \int_{p_t^-} R^-(x, t) da(x) + \int_{\partial p_t^- \setminus (\partial S_t^- \cap \bar{P}_t)} k^-(x, t) \bar{v}_t^-(x) ds(x) + \int_{p_t^\partial} R^\partial(x, t) da(x) + \int_{\partial p_t^\partial \setminus ((\Gamma_t^+ \cup \Gamma_t^-) \cap \bar{P}_t)} k^\partial(x, t) \bar{v}_t^\partial(x) ds(x) \quad (3.20)$$

where $\Gamma_t^\pm \equiv \partial S_t^\pm \setminus F_t$ and $\bar{v}_t^\partial = \bar{v}_t^\partial(x)$ is the outward pointing unit normal on ∂p_t^∂ ($\bar{v}_t^\partial(x) \in T_x(\partial B_t \setminus (p_t^+ \cup p_t^-))$).

Note that for an interior subbody we have $p_t^\partial = \emptyset$, $\partial S_t^+ \cap \bar{P}_t = \partial S_t^- \cap \bar{P}_t = F_t \cap \bar{P}_t$ and thus (3.18) - (3.20) include (3.2) - (3.4) as a special case.

We state the

Master balance law (Extended form): With B , P and S defined by (3.18) - (3.20) we have

$$\frac{d}{dt} B(P, t) - P(P, t) - S(P, t) = 0 \quad (Mb1)$$

for all subbodies (interior or at the boundary) belonging to B1, B2 or B3 at time t .

Proposition 3.5 (Local balance law 4)

$$\dot{\Psi}^\partial + \Psi^\partial \operatorname{div}_\Delta \dot{X}^\partial - \Pi^\partial - R^\partial + K \bar{n} - \operatorname{div}_\Delta k^\partial = 0 \quad (\text{Lb1 4})$$

$\partial B_t \setminus (S_t^+ \cup S_t^-) \ni x = X^\partial(X, t)$, $X \in \partial B \setminus \underline{S}_t$ (a boundary material point).

Proof: Analogous to the proof of Proposition 3.3. $\square$

Proposition 3.6 (Local balance law 5)

$$k^\pm \bar{v}^\pm + (k^\partial)^\pm \bar{v}^\partial = 0 \quad (\text{Lb1 5})$$

$\Gamma_t^\pm \ni x = X^\pm(X, t)$, $X \in \Gamma_t \equiv \partial S_t \setminus \underline{F}_t$. Here $(k^\partial)^\pm$ denotes the limits when $X \in \partial B \setminus \underline{S}_t$ approaches a point on Γ_t from "above" and "below" (cf. $F2^\partial$).

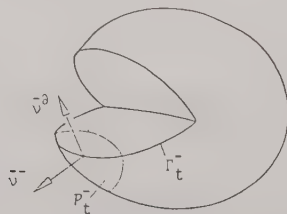


Figure 3.6

Proof: Let P be a cracked subbody at the boundary of B at time t . Consider the part P^- (see the figure above!). From (Mb1) we have

$$\begin{aligned} & \frac{d}{dt} \left(\int_{P_t^-} \Phi \, dv(x) + \int_{P_t^-} \Psi^- \, da(x) + \int_{P_t^-} \Psi^\partial \, da(x) \right) - \int_{P_t^-} \Pi \, dv(x) - \int_{P_t^-} \Pi_S^- \, da(x) - \\ & \int_{P_t^-} \Pi^\partial \, da(x) - \int_{\Sigma_t \cap P_t^-} \Pi_\Sigma^- \, da(x) - \int_{P_t^-} Q \, dv(x) - \int_{\partial P_t^- \setminus (P_t^- \cup \underline{P}_t^-)} K \bar{n} \, da(x) - \\ & \int_{P_t^-} R^- \, da(x) - \int_{\partial P_t^- \setminus (\Gamma_t^- \cap \bar{P}_t^-)} k^- \bar{v}^- \, ds(x) - \int_{P_t^-} R^\partial \, da(x) - \\ & \int_{\partial P_t^- \setminus (\Gamma_t^- \cap \bar{P}_t^-)} k^\partial \bar{v}^\partial \, ds(x) = 0. \end{aligned} \quad (3.21)$$

where $p_t^\partial \equiv \chi^\partial(\partial B \cap \bar{P}^-)$. It is a simple matter to show that (3.21) is equivalent to

$$\begin{aligned} & \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{X} - \Pi - Q - \operatorname{div} K) dv(x) - \int_{\Sigma_t \cap \bar{P}_t^-} ([\Phi \omega + K \bar{M}] + \Pi_\Sigma) da(x) + \\ & \int_{P_t} (\dot{\Psi}^- + \Psi^- \operatorname{div}_\Delta \dot{X}^- - \Pi_S^- - R^- + (K)^- \bar{m}^- - \operatorname{div}_\Delta k^-) da(x) + \int_{P_t} (\dot{\Psi}^\partial + \Psi^\partial \operatorname{div}_\Delta X^\partial \\ & - \Pi^\partial - R^\partial + K \bar{n} - \operatorname{div}_\Delta k^\partial) da(x) + \int_{\Gamma_t \cap \bar{P}_t^-} (k^- \bar{v}^- + (k^\partial)^- \bar{v}^\partial) ds(x) = 0 \end{aligned}$$

and by invoking the local balance laws 1 - 4 we find that this is equivalent to

$$\int_{\Gamma_t \cap \bar{P}_t^-} (k^- \bar{v}^- + (k^\partial)^- \bar{v}^\partial) ds(x) = 0 \quad (3.22)$$

and this proves the Proposition. $\square$

Remark: Note that $(k^\partial)^\pm \bar{v}^\partial \in L^1(\Gamma_t^\pm)$. This follows from (Lb1 5) and $\mathcal{G}^0$.

Proposition 3.6 (The reduced master balance law, extended form)

$$\begin{aligned} & \frac{d}{dt} \int_{P_t} \Phi dv(x) - \int_{P_t} (\Pi + Q) dv(x) - \int_{\partial P_t} K \bar{n} da(x) - \int_{\Sigma_t \cap P_t} \Pi_\Sigma da(x) + \\ & \frac{d}{dt} \int_{P_t} \Psi^\partial da(x) - \int_{\partial P_t} (\Pi^\partial + R^\partial - K \bar{n}) da(x) - \int_{\partial P_t} k^\partial \bar{v}^\partial ds(x) + \\ & \int_{F_t \cap \bar{P}_t} (\Psi c + k \bar{v}[-\Pi_F]) ds(x) = 0 \end{aligned} \quad (\text{Rmb1})_e$$

for all subbodies P (interior or at the boundary) belonging to any of the classes B1, B2 or B3 at time t .

Proof: Analogous to the proof of Proposition 3.4. $\square$

Theorem 3.1 is extended to

Theorem 3.2 The master balance law (in its extended form) is satisfied if and only if the local balance laws 1 - 5 and the reduced master balance law (in its extended form) are satisfied.

The fields

$$\Phi, \Pi, Q, K, \Psi^\pm, \Pi_S^\pm, R^\pm, k^\pm, \Pi_\Sigma, \Pi_F, \Psi^\partial, \Pi^\partial, R^\partial \text{ and } k^\partial$$

are said to be balanced if they satisfy the local balance laws 1 - 5 and the reduced balance law.

Remark: An extremely simplified model of the boundary ∂B is obtained if we put $\Psi^\partial \equiv 0$, $k^\partial \equiv 0$. Then the local balance laws 4 and 5 reduce to

$$-\Pi^\partial - R^\partial + K\bar{n} = 0 \quad \text{and} \quad k^\pm \bar{v}^\pm = 0$$

respectively. The reduced master balance law will have the form (Rmb1) (for all subbodies, interior or at the boundary).

3.2 The referential description

In the balance laws derived in the previous section we have taken the spatial point x (together with t) as the independent variable. Sometimes it is more convenient to use the referential point X instead of x . The local balance laws then transform to their referential forms. This is more or less a routine matter, but, for the sake of completeness, we collect, in this section, all the relevant formulas.

Consider the restriction of a fracture-deformation χ (the dependence on time is suppressed for the moment) to a non-cracking subbody P , i.e. a regular deformation $\chi: \bar{P} \rightarrow \chi(\bar{P})$.

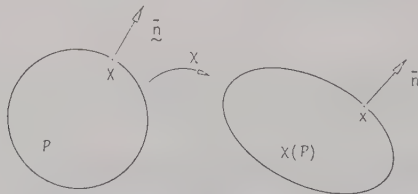


Figure 3.7

Let $\tilde{n}$ denote the outward pointing unit normal vector on ∂P at X and let $\bar{n}$ denote the corresponding vector on $\partial\chi(P)$ at $\chi(X)$. These vectors are

related by the well-known formula

Lemma 3.1

$$\bar{n}(X) = A^{-1} J F^{-T} \bar{n}(X) \quad (3.23)$$

where A is the area-ratio associated with the restriction of X to the surface ∂P , i.e. let $F_T(X)$ denote the restriction of $F(X)$ to T_X (the tangent-space of ∂P at X) then $A(X) \equiv (\det(F_T^T(X) F_T(X)))^{\frac{1}{2}}$ (cf. section 2.1).

Proof: Let $(\bar{e}_1, \bar{e}_2)$ be an orthonormal basis on T_X with positive orientation. We have (at the point X)

$$J F^{-T} \bar{n} = J F^{-T} (\bar{e}_1 \times \bar{e}_2) = F_T \bar{e}_1 \times F_T \bar{e}_2 = \bar{n} |F_T \bar{e}_1 \times F_T \bar{e}_2|$$

where we have used formula (1.5). But

$$\begin{aligned} |F_T \bar{e}_1 \times F_T \bar{e}_2|^2 &= (F_T \bar{e}_1 \cdot F_T \bar{e}_1)(F_T \bar{e}_2 \cdot F_T \bar{e}_2) - (F_T \bar{e}_1 \cdot F_T \bar{e}_2)^2 = \\ &= (\bar{e}_1 \cdot F_T^T F_T \bar{e}_1)(\bar{e}_2 \cdot F_T^T F_T \bar{e}_2) - (\bar{e}_1 \cdot F_T^T F_T \bar{e}_2)^2 = \det(F_T^T F_T) = A^2 \end{aligned}$$

and this proves the Lemma. $\square$

Remark: We have the following alternative representation of the area-ratio

$$A = J(\bar{n} \cdot \underline{C}^{-1} \bar{n})^{\frac{1}{2}} = J(\bar{n} \cdot \underline{B} \bar{n})^{-\frac{1}{2}}$$

where $\underline{C}$, $\underline{B}$ are the right and left Cauchy-Green tensors respectively.

Lemma 3.2

$$\text{Div}(J F^{-T}) = \bar{0} \quad (3.24)$$

at a regular bulk-material point.

Proof: Let P non-cracking subbody. From the divergence theorem and (3.23) we get

$$\bar{0} = \int_{\partial X(P)} \bar{n} da(X) = \int_{\partial P} J F^{-T} \bar{n} da(X) = \int_P \text{Div}(J F^{-T}) dv(X)$$

$$^1 (\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c})$$

and this proves the Lemma. $\square$

Remark: The dual formula to (3.24) may be written $\text{div}(J^{-1} \underline{F}^T) = \bar{0}$.

The Piola-fields Φ_p , Π_p , Q_p and K_p , associated with Φ , Π , Q and K respectively, are defined by (cf. section 2.2)

$$\Phi_p \equiv J \Phi, \quad \Pi_p \equiv J \Pi, \quad Q_p \equiv J Q \quad (3.25)$$

and

$$K_p \equiv \begin{cases} J \underline{F}^{-1} K, & \text{if } K(X) \in V \\ J K \underline{F}^{-T}, & \text{if } K(X) \in \text{End}(V) \end{cases} \quad (3.26)$$

Proposition 3.7 We have the relations

$$\int_P \Phi_p(X) dv(X) = \int_{X(P)} \Phi(x) dv(x) \quad (3.27)$$

$$\int_{\partial P} K_p(X) \bar{n}(X) da(X) = \int_{\partial X(P)} K(x) \bar{n}(x) da(x)$$

and the Piola-identity

$$\text{Div } K_p = J \text{div } K \quad (3.28)$$

Proof: Formulas (3.27) are direct consequences of (3.25), (3.26), (3.23) and the change of variable formula. For the proof of the Piola-identity we assume that $K(X) \in V$ (the case $K(X) \in \text{End}(V)$ is equally simple). A straightforward calculation gives

$$\begin{aligned} \text{Div } K_p &= \text{Div}(J \underline{F}^{-1} K) = (\text{Div}(J \underline{F}^{-T})) \cdot K + \text{tr}(J \underline{F}^{-1} \text{Grad } K) = J \text{tr}(\text{grad } K) = \\ &= J \text{div } K \end{aligned}$$

where we have used (3.24) and the relation $\text{Grad } K = (\text{grad } K) \underline{F}$. $\square$

We now turn to the balance laws.

Proposition 3.8 The local balance law 1 is equivalent to

$$\dot{\Phi}_p - \Pi_p - Q_p - \text{Div } K_p = 0 \quad (\text{Lb1 1})_p$$

at X , a regular bulk-material point.

Proof: By using (2.46)₁, (3.25) and (3.28) we get

$$\dot{\Phi}_p - \Pi_p - Q_p - \text{Div } K_p = J(\dot{\Phi} + \Phi \text{div } \dot{X} - \Pi - Q - \text{div } K)$$

and the Proposition follows from the fact that $J(X, t) > 0$. $\square$

To the singular-surface-field Π_Σ we associate the Piola-field $\Pi_{\Sigma p}$ defined by

$$\Pi_{\Sigma p} \equiv A_\Sigma \Pi_\Sigma \quad (3.29)$$

where A_Σ is the area-ratio associated with the restriction of X to Σ .

Proposition 3.9 The local balance law 2 is equivalent to

$$[\Phi_p \underline{\omega} + K_p \underline{\bar{M}}] + \Pi_{\Sigma p} = 0 \quad (\text{Lb1 2})_p$$

at X ; a singular bulk-material point ($\underline{\omega}$ and $\underline{\bar{M}}$ are introduced in section 2.4).

Proof: By using the relations (cf. Lemma 2.13 in section 2.4)

$$\omega^\pm = \underline{\omega} A_\Sigma^{-1} (J)^\pm, \quad \bar{M} = A_\Sigma^{-1} (J)^\pm ((F)^\pm)^{-T} \underline{\bar{M}},$$

(3.25), (3.26) and (3.29) we get

$$[\Phi_p \underline{\omega} + K_p \underline{\bar{M}}] + \Pi_{\Sigma p} = A_\Sigma ([\Phi \omega + K \bar{M}] + \Pi_\Sigma)$$

and the Proposition follows from the fact that $A_{\Sigma, t}(X) > 0$. $\square$

Let X be a fracture-deformation. Consider the restriction of $X^\pm$ to the surface $p \equiv \Sigma \cap P$ where P is a cracked subbody, i.e. the regular deformation

$$\chi^\pm : p \rightarrow p^\pm \equiv X^\pm(p)$$

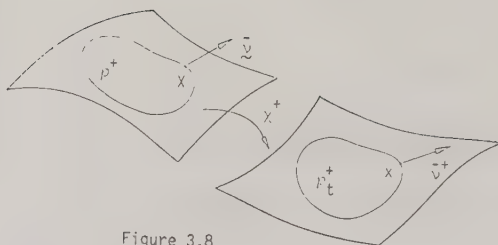


Figure 3.8

Let $\tilde{v}$ denote the outward pointing unit normal vector on ∂p at X and let $\tilde{v}^{\pm}$ denote the corresponding vector on $\partial p^{\pm}$ (see the figure above!). These vectors are related by

Lemma 3.3

$$\tilde{v}^{\pm}(X^{\pm}(X)) = (L^{\pm})^{-1} A^{\pm} (F^{\pm})^{-T} \tilde{v}(X) \quad (3.30)$$

where $L^{\pm}$ is the length-ratio associated with the restriction of $\chi^{\pm}$ to the curve ∂p .

Proof: Let $\tilde{e}$ denote a unit tangent vector on ∂p (we may take $\tilde{e} = \tilde{m} \times \tilde{v}$). We have

$$F^{\pm} \tilde{e} = L^{\pm} \tilde{e}^{\pm} \quad (3.31)$$

where $\tilde{e}^{\pm}$ are unit tangent vectors on $\partial p^{\pm}$. Furthermore we have

$$\tilde{e} \cdot (F^{\pm})^T \tilde{v}^{\pm} = \tilde{v}^{\pm} \cdot F^{\pm} \tilde{e} = L^{\pm} \tilde{v}^{\pm} \cdot \tilde{e}^{\pm} = 0$$

Thus, for some $q^{\pm} > 0$, we have

$$(F^{\pm})^T \tilde{v}^{\pm} = q^{\pm} \tilde{v} \quad (3.32)$$

As in the proof of Lemma 3.1 we may write

$$A^{\pm} \equiv \det \underline{U}^{\pm} = |F^{\pm} \tilde{v} \times F^{\pm} \tilde{e}| = |(\tilde{v}^{\pm} \cdot F^{\pm} \tilde{v}) \tilde{v}^{\pm} \times L^{\pm} \tilde{e}^{\pm}| = |L^{\pm} (\tilde{v} \cdot (F^{\pm})^T \tilde{v}^{\pm}) \tilde{m}^{\pm}| = L^{\pm} q^{\pm} \quad (3.33)$$

where we have used (3.31) and (3.32). The Lemma now follows from (3.32) and (3.33). $\square$

Remark: We have the following alternative representation for the length-ratio

$$L^{\pm} = A^{\pm} (\tilde{\gamma} \cdot (\underline{C}^{\pm})^{-1} \tilde{\gamma})^{\frac{1}{2}} = A^{\pm} (\tilde{\nu}^{\pm} \cdot \underline{B}^{\pm} \tilde{\nu}^{\pm})^{-\frac{1}{2}}$$

where $\underline{C}^{\pm}$, $\underline{B}^{\pm}$ are the right and left Cauchy-Green tensors on $\tilde{\Sigma}$, $\underline{C}^{\pm} \equiv (\underline{F}^{\pm})^T \underline{F}^{\pm}$, $\underline{B}^{\pm} \equiv \underline{F}^{\pm} (\underline{F}^{\pm})^T$.

Lemma 3.4

$$\text{Div}_{\Delta} (A^{\pm} (\underline{F}^{\pm})^{-T}) = A^{\pm} 2 M^{\pm} \bar{m}^{\pm} \quad (3.34)$$

at a crack-surface material point ($M^{\pm}$ are the mean-curvatures defined in (2.24)).

Proof: Let P be a cracked subbody and take $p \equiv \tilde{\Sigma} \cap P$. From the divergence theorem and (3.30) we get

$$\begin{aligned} \int_p \text{Div}_{\Delta} (A^{\pm} (\underline{F}^{\pm})^{-T}) da(X) &= \int_{\partial p} A^{\pm} (\underline{F}^{\pm})^{-T} \tilde{\gamma} ds(X) = \int_{\partial p^{\pm}} \tilde{\nu}^{\pm} ds(x) = \\ \int_{p^{\pm}} \text{div}_{\Delta} \underline{l}_{T^{\pm}} da(x) &= \int_p 2 M^{\pm} \bar{m}^{\pm} A^{\pm} da(X) \end{aligned} \quad (3.35)$$

where we, in the last step, have used the relation (see Appendix A.6)

$$\text{div}_{\Delta} \underline{l}_{T^{\pm}} = 2 M^{\pm} \bar{m}^{\pm} \quad (3.36)$$

From (3.35) we have

$$\int_p (\text{Div}_{\Delta} (A^{\pm} (\underline{F}^{\pm})^{-T}) - A^{\pm} 2 M^{\pm} \bar{m}^{\pm}) da(X) = 0$$

and this proves the Lemma. $\square$

Remark: The dual formula to (3.34) may be written

$$\text{div}_{\Delta} ((A^{\pm})^{-1} (\underline{F}^{\pm})^T) = (A^{\pm})^{-1} 2 M_{\tilde{\Sigma}}^{\pm}$$

¹ $\text{Div}_{\Delta} (A^{\pm} (\underline{F}^{\pm})^{-T}) \equiv \text{Div}_{\Delta} (\underline{I}^{\pm} A^{\pm} (\underline{F}^{\pm})^{-T})$, see Appendix A.6.

The Piola-fields $\psi_p^\pm$, $\pi_{Sp}^\pm$, $R_p^\pm$ and $k_p^\pm$, associated with $\Psi^\pm$, $\Pi_S^\pm$, $R^\pm$ and respectively, are defined by

$$\psi_p^\pm \equiv A^\pm \Psi^\pm, \quad \pi_{Sp}^\pm \equiv A^\pm \Pi_S^\pm, \quad R_p^\pm \equiv A^\pm R^\pm \quad (3.37)$$

and

$$k_p^\pm = \begin{cases} A^\pm (E^\pm)^{-1} k^\pm, & \text{if } k^\pm(X) \in T_{X^\pm}^\pm(X) \\ A^\pm I^\pm k^\pm (E^\pm)^{-T}, & \text{if } k^\pm(X) \in \text{End}(T_{X^\pm}^\pm(X)) \\ A^\pm k^\pm (E^\pm)^{-T}, & \text{if } k^\pm(X) \in L(T_{X^\pm}^\pm(X), V) \end{cases} \quad (3.38)$$

Proposition 3.10 We have the relations

$$\int_p \psi_p^\pm(X) da(X) = \int_{p^\pm} \Psi^\pm(x) da(x) \quad (3.39)$$

$$\int_{\partial p} k_p^\pm(X) \bar{\omega}(X) ds(X) = \int_{\partial p^\pm} k^\pm(x) \bar{\omega}^\pm(x) ds(x)$$

and the Piola-identity

$$\text{Div}_\delta k_p^\pm = A^\pm \text{div}_\delta k^\pm \quad (3.40)$$

Proof: Formulas (3.39) are direct consequences of (3.37), (3.38), (3.30) and the change of variable formula. For the proof of the Piola-identity we assume that $k^\pm(X) \in \text{End}(T_{X^\pm}^\pm(X))$ (the other two cases are equally simple). A straightforward calculation gives ($\bar{a}$ is a fixed vector in V)

$$\begin{aligned} \bar{a} \cdot \text{Div}_\delta k_p^\pm &= \text{Div}_\delta ((k_p^\pm)^T \bar{a}) = \text{Div}_\delta (A^\pm (E^\pm)^{-1} (k^\pm)^T E^\pm \bar{a})^1 \\ &= (\text{Div}_\delta (A^\pm (E^\pm)^{-1})) \cdot (k^\pm)^T E^\pm \bar{a} + \text{tr}(A^\pm (E^\pm)^{-1} E^\pm \text{Grad}_\delta ((k^\pm)^T E^\pm \bar{a})) = \\ &= A^\pm \text{tr}((E^\pm)^{-1} E^\pm \text{grad}_\delta ((k^\pm)^T E^\pm \bar{a}) E^\pm) = A^\pm \text{div}_\delta ((k^\pm)^T E^\pm \bar{a}) = A^\pm (\text{div}_\delta k^\pm) \cdot \bar{a} \end{aligned}$$

for all $\bar{a} \in V$, where we have used (3.34). $\square$

Remark: If $k^\pm(X) \in V$ or $\text{End}(V)$ then, if we introduce the definitions $k_p^\pm \equiv A^\pm (E^\pm)^{-1} E^\pm k^\pm$ and $k_p^\pm \equiv A^\pm k^\pm I^\pm (E^\pm)^{-T}$ respectively, formula (3.39)₂ is still valid but the Piola-identity is changed for

¹ (A.23)₆

$$\text{Div}_\Delta k_p^\pm = A^\pm (\text{div}_\Delta k^\pm + 2 M^\pm k^\pm \bar{m}^\pm)$$

Proposition 3.11 The local balance law 3 is equivalent to

$$\dot{\Psi}_p^\pm - \pi_{Sp}^\pm - R_p^\pm \mp (K_p)^\pm \bar{m} - \text{Div}_\Delta k_p^\pm = 0 \quad (\text{Lb1 3})_p$$

at X , a crack-surface material point.

Proof: By using (2.56)₁, (3.37), (3.40) and the relation

$$(K_p)^\pm \bar{m} = \mp A^\pm (K)^\pm \bar{m}^\pm \quad (3.41)$$

we get

$$\begin{aligned} \dot{\Psi}_p^\pm - \pi_{Sp}^\pm - R_p^\pm \mp (K_p)^\pm \bar{m} - \text{Div}_\Delta k_p^\pm &= A^\pm (\dot{\Psi}^\pm + \Psi^\pm \text{div}_\Delta \dot{x}^\pm - \pi_S^\pm + (K)^\pm \bar{m}^\pm - \\ &\quad \text{div}_\Delta k^\pm) \end{aligned}$$

and the Proposition follows from the fact that $A^\pm(X, t) > 0$. $\square$

Analogous to Proposition 3.11 we have

Proposition 3.12 The local balance law 4 is equivalent to

$$\dot{\Psi}_p^\partial - \pi_p^\partial - R_p^\partial + K_p \bar{n} - \text{Div}_\Delta k_p^\partial = 0 \quad (\text{Lb1 4})_p$$

$X \in \partial B \setminus \underline{S}_t$. Here $\dot{\Psi}_p^\partial$, π_p^∂ , R_p^∂ and k_p^∂ are Piola-fields corresponding to the boundary-fields introduced in section 3.1.

Proposition 3.13 The local balance law 5 is equivalent to

$$k_p^\pm \bar{\nu} \mp (k_p^\partial)^\pm \bar{\nu}^\partial = 0 \quad (\text{Lb1 5})_p$$

$X \in \underline{\Gamma}_t$. Here $\bar{\nu}^\partial \equiv \bar{n} \times \bar{e}$ where $\bar{e}$ is a unit tangent vector-field on $\underline{\Gamma}_t$ in co-ordination with the orientation $\bar{m}$ on $\underline{S}_t$. See the figure below.

Proof: This is an immediate consequence of Lemma 3.3 and the definition of the Piola-fields $k_p^\pm$ and k_p^∂ . $\square$

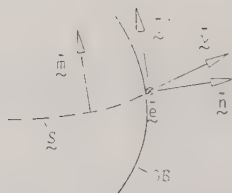


Figure 3.9

To the crack-front field Π_F (introduced in section 3.1) we associate the Piola-field Π_{Fp} defined by

$$\Pi_{Fp} = L \Pi_F \quad (3.42)$$

where L is the length-ratio associated with the restriction of $\chi^\pm$ to the crack-front γ_C .

Proposition 3.14 The reduced master balance law is equivalent to

$$\begin{aligned} & \frac{d}{dt} \int_P \Phi_p \, dv(X) - \int_P (\Pi_p + Q_p) \, dv(X) - \int_{\partial P} K_p \tilde{n} \, da(X) + \int_{\Sigma_t \cap P} [K_p] \tilde{n} \, da(X) - \\ & \int_{\Sigma_t \cap P} \Pi_{\Sigma p} \, da(X) + \frac{d}{dt} \int_{p^a} \Psi_p^a \, da(X) - \int_{p^a} (\Pi_p^a + R_p^a - K_p \tilde{n}) \, da(X) - \\ & \int_{\partial p^a} k_p^a \tilde{v}^a \, ds(X) + \int_{\Sigma_t \cap p^a} [k_p^a] \tilde{v}^a \, ds(X) + \int_{\Sigma_t \cap \bar{P}} (I \Psi_p \tilde{c} + k_p \tilde{v} [- \Pi_{Fp}) \, dv(X) \\ & = 0 \end{aligned} \quad (Rmb1)_p$$

for all subbodies P . Here $p^a \equiv \partial B \cap \bar{P}$.

Proof: From (3.30), (3.37), (3.38), (3.42) and the relation (cf. (2.29))

$$\tilde{c} = c^\pm L (A^\pm)^{-1} \quad (3.43)$$

we obtain

$$I \Psi_p \tilde{c} + k_p \tilde{v} [- \Pi_{Fp} = L (I \Psi c + k \tilde{v} [- \Pi_F) \quad (3.44)$$

and this proves

$$\int_{\tilde{F}_t \cap P} (\Psi_P \tilde{c} + k_P \tilde{v} - \pi_{FP}) ds(X) = \int_{F_t \cap \tilde{P}_t} (\Psi_c + k \tilde{v} - \pi_F) ds(x)$$

The rest of the proof is a simple consequence of (3.27) and (3.39). $\square$

4. THE RELEASE RATE

Throughout this chapter we assume that the fields

$$\Phi, \Pi, Q, K, \Psi^\pm, \Pi_\Sigma^\pm, R^\pm, k^\pm, \pi_\infty, \pi_F, \Psi^\partial, \Pi^\partial, R^\partial \text{ and } k^\partial$$

are balanced. The reduced master balance law may be written

$$\int_{F_t \cap \bar{P}_t} (J \Psi c + k \bar{\nu} [-\pi_F] ds(x)) = R(P, t) \quad (4.1)$$

where

$$\begin{aligned} R(P, t) = & \int_{\partial P_t} K \bar{n} da(x) + \int_{P_t} (\Pi + Q) dv(x) - \frac{d}{dt} \int_{P_t} \Phi dv(x) + \int_{\Sigma_t \cap P_t} \pi_\Sigma da(x) + \\ & \int_{\partial P_t} k^\partial \bar{\nu}^\partial ds(x) + \int_{P_t} (\Pi^\partial + R^\partial - K \bar{n}) da(x) - \frac{d}{dt} \int_{P_t} \Psi^\partial da(x) \end{aligned} \quad (4.2)$$

is called the release rate of the balanced quantity. We have the following simple observation

Proposition 4.1

$$R(P_1, t) = R(P_2, t) \quad (4.3)$$

if P_1 and P_2 are cracking subbodies at time t satisfying $\bar{F}_t \cap P = \bar{F}_t \cap P_2$. Furthermore

$$R(P, t) = 0 \quad (4.4)$$

if P is a non-cracking or cracked subbody at time t .

Proof: This is a trivial consequence of the reduced master balance law.

□

Remark: Note that, in the case of a non-cracking or cracked subbody, (4.1) is not an independent condition. It follows from the local balance laws 1 - 5.

It is possible to eliminate the fields $\Pi, Q, \pi_\Sigma, \Pi^\partial$ and R^∂ from formula (4.2) and express the release rate in terms of Φ, K, Ψ^∂ and k^∂ only.

Proposition 4.2

$$\begin{aligned}
R(P, t) = & \int_{\partial P_t} K \bar{n} da(x) + \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x} - \operatorname{div} K) dv(x) - \frac{d}{dt} \int_{P_t} \Phi dv(x) - \\
& \int_{\Sigma_t \cap P_t} [\Phi w + K \bar{M}] da(x) + \int_{\partial P_t} k^a \bar{v}^a ds(x) + \int_{P_t} (\dot{\Psi}^a + \Psi^a \operatorname{div}_\Delta \dot{x}^a - \\
& \operatorname{div}_\Delta k^a) da(x) - \frac{d}{dt} \int_{P_t} \Psi^a da(x) \quad (4.5)
\end{aligned}$$

Proof: This follows from the local balance laws 1, 2 and 4. $\square$

If $\Psi^a \equiv 0$, $k^a \equiv 0$ or if P is interior ($P_t^a \equiv \emptyset$) then

$$\begin{aligned}
R(P, t) = & \int_{\partial P_t} K \bar{n} da(x) + \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x} - \operatorname{div} K) dv(x) - \frac{d}{dt} \int_{P_t} \Phi dv(x) - \\
& \int_{\Sigma_t \cap P_t} [\Phi w + K \bar{M}] da(x) \quad (4.6)
\end{aligned}$$

Proposition 4.3 If $\Psi^a \equiv 0$, $k^a \equiv 0$ and if we have the relations

$$\int_{\partial P_t} K \bar{n} da(x) = \int_{P_t} \operatorname{div} K dv(x) + \int_{\Sigma_t \cap P_t} [K] \bar{M} da(x) \quad (4.7)$$

and

$$\frac{d}{dt} \int_{P_t} \Phi dv(x) = \int_{P_t} (\dot{\Phi} + \Phi \operatorname{div} \dot{x}) dv(x) - \int_{\Sigma_t \cap P_t} [\Phi w] da(x) \quad (4.8)$$

then the reduced master balance law is equivalent to the local condition

$$[\Psi] c + k \bar{v}[-\Pi_F] = 0, \quad x \in F_t \quad (4.9)$$

Proof: From (4.1), (4.6) - (4.8) we get

$$\int_{F_t \cap P_t} ([\Psi] c + k \bar{v}[-\Pi_F]) ds(x) = 0$$

for any subbody P and this proves the Proposition. $\square$

Remark: The validity of formula (4.8), apart from a minor modification

due to the presence of the singular surface Σ , has been discussed in section 2.2 (Propositions 2.5 and 2.6). Formula (4.7) expresses the divergence theorem for the field K and the region P_t . Note that in general P_t need not be a regular region and $\text{div} K$ need not be integrable.

The representation of the release rate, given in (4.5), is of a rather complicated nature. A main task will be to try to find simplified and more transparent alternatives to (4.5). In order to bring about these simplifications one may have to introduce further a priori assumptions on χ , Φ , K , Ψ^{∂} and k^{∂} . The crucial point is to find assumptions that are not too strong so that "realistic fracture-motions" are not ruled out from the start. The next section is devoted to a discussion on an important re-representation of the release rate called the residual form.

4.1 The residual form

Throughout this section we assume that $\Psi^{\partial} \equiv 0$ and $k^{\partial} \equiv 0$.

As a follow-up of the kinematical discussions in section 2.3 we have

Proposition 4.4 If

$$\begin{aligned} \frac{d}{dt} \int_{P_t} \Phi \, dv(x) &= \int_{P_t} (\Phi' + \Phi \, \text{div} \, x') \, dv(x) - \int_{\partial P_t} \Phi (x' - \dot{x}) \cdot \bar{n} \, da(x) - \\ &\quad \int_{\Sigma_t \cap P_t} [\Phi (\bar{w} - x')] \cdot \bar{M} \, da(x) \end{aligned} \quad (4.10)$$

then

$$\begin{aligned} R(P, t) &= \int_{\partial P_t} (K + \Phi \otimes (x' - \dot{x})) \bar{n} \, da(x) - \int_{\Sigma_t \cap P_t} [K + \Phi \otimes (x' - \dot{x})] \bar{M} \, da(x) - \\ &\quad \int_{P_t} \text{div} (K + \Phi \otimes (x' - \dot{x})) \, dv(x) \end{aligned} \quad (4.11)$$

Proof: From (4.6) and (4.10) we get

$$\begin{aligned} R(P, t) &= \int_{\partial P_t} (K + \Phi \otimes (x' - \dot{x})) \bar{n} \, da(x) - \int_{\Sigma_t \cap P_t} [K + \Phi \otimes (x' - \dot{x})] \bar{M} \, da(x) - \\ &\quad \int_{P_t} (\text{div} K + \Phi' + \Phi \, \text{div} \, x' - \dot{\Phi} - \Phi \, \text{div} \, \dot{x}) \, dv(x) \end{aligned}$$

But, from (2.91), we have

$$\Phi' + \Phi \operatorname{div} x' - \dot{\Phi} - \Phi \operatorname{div} \dot{x} = \operatorname{div}(\Phi \otimes (x' - \dot{x}))$$

and this proves the Proposition. $\square$

Remark 1: The validity of formula (4.10), apart from a minor modification due to the presence of the singular surface Σ , has been discussed in section 2.3 (Propositions 2.9 and 2.11. The additional assumption needed: $[\Phi(\bar{w} - x') \cdot \bar{M}] \in L^1(\Sigma_t \cap P_t)$ uniformly in time).

Remark 2: From (4.11) it follows that if the divergence theorem is valid for the field $K + \Phi \otimes (x' - \dot{x})$ and the region P_t then $R(P, t) = 0$.

Remark 3: In the referential description (4.11) may be written

$$\begin{aligned} R(P, t) = & \int_{\partial P} (K_p + \Phi_p \otimes \bar{\zeta}) \bar{n} da(X) - \int_{\Sigma_t \cap P} [K_p + \Phi_p \otimes \bar{\zeta}] \bar{m} da(X) - \\ & \int_{\Sigma_t \cap P} [K_p + \Phi_p \otimes \bar{\zeta}] \bar{M} da(X) - \int_P \operatorname{Div}(K_p + \Phi_p \otimes \bar{\zeta}) dv(X) \quad (4.12) \end{aligned}$$

Theorem 4.1 If formula (4.10) is valid then we have the following residual form of the release rate

$$R(P, t) = \lim_{R \rightarrow 0} \int_{Tu_t(R) \cap P_t} (K + \Phi \otimes (x' - \dot{x})) \bar{e}_r da(x) \quad (\text{Res})$$

where $Tu_t(R)$ is a circular tube surrounding the crack-front in the spatial description and $\bar{e}_r$ is the outward pointing (relative to $\operatorname{int}(Tu_t(R))$) unit normal vector on $Tu_t(R)$ (cf. section 2.3).

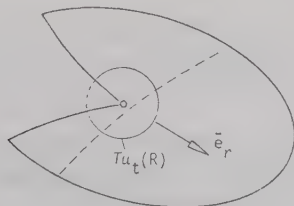


Figure 4.1

Proof: Put $P_t(R) \equiv P_t \setminus \text{int}(Tu_t(R))$ and $\Delta_t(R) \equiv \partial P_t \cap \text{int}(Tu_t(R))$. Then

$$\lim_{R \rightarrow 0} \text{volume}(P_t \setminus P_t(R)) = 0, \quad \lim_{R \rightarrow 0} \text{area}(\Delta_t(R)) = 0 \quad (4.13)$$

and the divergence theorem gives (see Figure 4.1!)

$$\begin{aligned} \int_{Tu_t(R) \cap P_t} (K + \Phi \otimes (x' - \dot{x})) \bar{e}_r da(x) &= \int_{\partial P_t \setminus \Delta_t(R)} (K + \Phi \otimes (x' - \dot{x})) \bar{n} da(x) - \\ \int_{\Sigma_t \cap P_t(R)} [K + \Phi \otimes (x' - \dot{x})] \bar{M} da(x) &- \int_{P_t(R)} \text{div}(K + \Phi \otimes (x' - \dot{x})) dv(x) \end{aligned} \quad (4.14)$$

From (4.11) and (4.13) we conclude that the right membrum in (4.14) converges to $R(P, t)$ as $R \rightarrow 0$. This proves the Proposition. $\square$

Remark 1: It is clear from the proof of Theorem 4.1 that if $T\delta_t(\delta)$, $\delta > 0$ is any family of tubular surfaces surrounding the crack-front in the spatial description and satisfying

$$\lim_{\delta \rightarrow 0} \text{rad}(T\delta_t(\delta)) = 0$$

(rad is defined in (2.67)₁) then

$$R(P, t) = \lim_{\delta \rightarrow 0} \int_{T\delta_t(\delta) \cap P_t} (K + \Phi \otimes (x' - \dot{x})) \bar{N} da(x)$$

where $\bar{N}$ is the outward pointing unit normal vector on $T\delta_t(\delta)$.

Remark 2: In the referential description (Res) may be written

$$R(P, t) = \lim_{R \rightarrow 0} \int_{\tilde{Tu}_t(R) \cap P} (K_p + \Phi_p \otimes \tilde{C}) \tilde{e}_r da(X) \quad (\text{Res})_p$$

where $\tilde{Tu}_t(R)$ is a circular tube surrounding the crack-front in the referential description and $\tilde{e}_r$ is the outward pointing unit normal vector on $\tilde{Tu}_t(R)$ (cf. section 2.3).

We will now approach the residual form from a somewhat different starting-point. Let P be a cracking subbody at time t . According to 1° there is an open time-interval $T \ni t$ such that the function

$$T \ni s \leadsto \int_P \Phi_p(X, s) dv(X) \quad (4.15)$$

is differentiable. Let $\tilde{T}_{u_s}(R)$, $s \in T^-$ be a moving circular tube accompanying the crack-front in the referential description. Put

$$\tilde{P}_s(R) \equiv P \setminus \text{int}(\tilde{T}_{u_s}(R)) \quad (4.16)$$

Proposition 4.5 If

$$\frac{d}{ds} \int_{\tilde{P}_s(R)} \Phi_p(X, s) dv(X) \big|_{s=t} \rightarrow \frac{d}{ds} \int_P \Phi_p(X, s) dv(X) \big|_{s=t} \quad (4.17)$$

as $R \rightarrow 0$, then

$$R(P, t) = \lim_{R \rightarrow 0} \int_{\tilde{T}_{u_t}(R) \cap P} (K_p + \Phi_p \otimes \tilde{c}) \cdot \tilde{e}_r da(X) \quad (\text{Res})_p$$

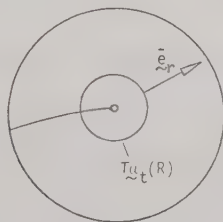


Figure 4.2

Proof: We have (cf. Proposition 2.14)

$$\begin{aligned} \frac{d}{ds} \int_{\tilde{P}_s(R)} \Phi_p dv(X) &= \int_{\tilde{P}_s(R)} \dot{\Phi}_p dv(X) - \int_{\tilde{T}_{u_s}(R) \cap P} \Phi_p \tilde{c} \cdot \tilde{e}_r da(X) - \\ &\quad \int_{\Sigma_s \cap P} [\Phi_p] \tilde{w} da(X) \end{aligned} \quad (4.18)$$

Proof: Put $\Delta_s^1 \equiv \partial P \cap \text{int}(\tilde{T}_{u_s}(R))$ and $\Delta_s^2 \equiv \Sigma_s \cap P \cap \text{int}(\tilde{T}_{u_s}(R))$. Then

$$\lim_{R \rightarrow 0} \text{area}(\Delta_s^i(R)) = 0, \quad i = 1, 2 \quad (4.19)$$

and the divergence theorem gives

$$\int_{\tilde{P}_s(R)} \text{Div } K_p dv(X) = \int_{\partial P \setminus \Delta_s^1(R)} K_p \tilde{n} da(X) - \int_{(\Sigma_s \cap P) \setminus \Delta_s^2(R)} [K_p] \tilde{m} da(X) -$$

$$\mathcal{T}_{u_s}(R) \cap P \int_P K_p \bar{e}_r da(x) - \int_{\Sigma_s \cap \mathcal{P}_s(R)} [K_p] \bar{M} da(X) \quad (4.20)$$

From (4.18) and (4.20) we get

$$\begin{aligned} \mathcal{T}_{u_t}(R) \cap P \int_P (K_p + \Phi_p \otimes \bar{c}) \bar{e}_r da(X) &= \int_{\partial P \setminus \Delta_t^1(R)} K_p \bar{n} da(X) - \\ (\Sigma_t \cap P) \setminus \Delta_t^2(R) \int_{\mathcal{P}_t(R)} [K_p] \bar{m} da(X) &- \int_{\mathcal{P}_t(R)} (\Phi_p - \text{Div } K_p) dv(X) - \frac{d}{ds} \int_{\mathcal{P}_s(R)} \Phi_p dv(X) \Big|_{s=t} - \\ \int_{\Sigma_t \cap \mathcal{P}_t(R)} [\Phi_p \bar{w} + K_p] \bar{M} da(X) &\quad (4.21) \end{aligned}$$

From (4.17), (4.19) and the fact that $\text{volume}(P \setminus \mathcal{P}_t(R)) \rightarrow 0$ as $R \rightarrow 0$ we conclude that the right membrum in (4.21) converges to $R(P, t)$ as $R \rightarrow 0$. This proves the Proposition. $\square$

Remark: The assumption in (4.17) was first introduced in Gurtin [12].

Note that if we, instead of (4.17), assume that

$$\frac{d}{ds} \int_{\mathcal{P}_s(R)} \Phi_p(X, s) dv(X)$$

converges as $R \rightarrow 0$, to a limit function uniformly in $s \in T^-$, then it follows that (4.15) is a differentiable function and that the limit in (4.17) is valid.

Proposition 4.6 If the limit in (4.17) is valid and if in addition

$$\begin{aligned} \text{Div}(K_p + \Phi_p \otimes \bar{c}) &\in L^1(P), \quad \Phi_p \bar{c} \cdot \bar{n} \in L^1(\partial P), \quad (\Phi_p)^* \bar{c} \cdot \bar{m} \in L^1(\Sigma_t \cap P) \\ \text{and } [K_p + \Phi_p \otimes \bar{c}] \bar{M} &\in L^1(\Sigma_t \cap P) \end{aligned} \quad (4.22)$$

then

$$\begin{aligned} R(P, t) &= \int_{\partial P} (K_p + \Phi_p \otimes \bar{c}) \bar{n} da(X) - \int_{\Sigma_t \cap P} [K_p + \Phi_p \otimes \bar{c}] \bar{m} da(X) - \\ &\int_{\Sigma_t \cap P} [K_p + \Phi_p \otimes \bar{c}] \bar{M} da(X) - \int_P \text{Div}(K_p + \Phi_p \otimes \bar{c}) dv(X) = \\ \lim_{\delta \rightarrow 0} \mathcal{T}_{\Delta_t}(\delta) \int_P (K_p + \Phi_p \otimes \bar{c}) \bar{n} da(X) &\quad (\text{Res})_p \end{aligned}$$

for any family of tubular surfaces $T_{\delta_t}(\delta)$, $\delta > 0$ surrounding the crack-front in the referential description and satisfying

$$\lim_{\delta \rightarrow 0} \text{rad}(T_{\delta_t}(\delta)) = 0 \quad (4.23)$$

($\bar{\mathbf{N}} = \bar{\mathbf{N}}_{t,\delta}(X)$ is the outward pointing unit normal vector on $T_{\delta_t}(\delta)$).

Proof: The divergence theorem gives (with notation as in the proof of Proposition 4.5)

$$\begin{aligned} \int_{T_{\delta_t}(R) \cap P} (K_p + \Phi_p \otimes \bar{\mathbf{C}}) \bar{\mathbf{e}}_r \, da(X) &= \int_{\partial P \setminus \Delta_t^1(R)} (K_p + \Phi_p \otimes \bar{\mathbf{C}}) \bar{\mathbf{n}} \, da(X) - \\ &(\bar{S}_t \cap P) \setminus \Delta_t^2(R) [K_p + \Phi_p \otimes \bar{\mathbf{C}}] \bar{\mathbf{m}} \, da(X) - \int_{\Sigma_t \cap P_t(R)} [K_p + \Phi_p \otimes \bar{\mathbf{C}}] \bar{\mathbf{M}} \, da(X) - \\ &\int_{P_t(R)} \text{Div}(K_p + \Phi_p \otimes \bar{\mathbf{C}}) \, dv(X) \end{aligned} \quad (4.24)$$

By taking limits in (4.23) (as $R \rightarrow 0$) and using (4.22) and Proposition 4.5 we obtain the first equality in (Res)_P (note that from 3° and Corollary 3.1 we have $K_p \bar{\mathbf{n}} \in L^1(\partial P)$ and $(K_p)^\pm \bar{\mathbf{m}} \in L^1(\bar{S}_t \cap P)$). The second equality is now obvious. $\square$

Remark 1: Note that if $\bar{S}_t \cap P$ is a plane surface then $(\Phi_p)^\pm \bar{\mathbf{C}} \cdot \bar{\mathbf{m}} = 0$ on $\bar{S}_t \cap P$.

Remark 2: It is a simple matter to demonstrate that (4.22) is equivalent to

$$\begin{aligned} \Phi' + \Phi \text{div } x' &\in L^1(P_t), \quad \Phi(x' - \dot{x}) \cdot \bar{\mathbf{n}} \in L^1(\partial P_t) \quad \text{and} \\ [\Phi(\bar{w} - x')] \cdot \bar{\mathbf{M}} &\in L^1(\Sigma_t \cap P_t) \end{aligned} \quad (4.25)$$

(cf. formula (4.10)).

Our assumption so far do not guarantee the existence of the separate limits

$$\lim_{R \rightarrow 0} \int_{T_{\delta_t}(R) \cap P} K_p \bar{\mathbf{e}}_r \, da(X), \quad \lim_{R \rightarrow 0} \int_{T_{\delta_t}(R) \cap P} \Phi_p \bar{\mathbf{C}} \cdot \bar{\mathbf{e}}_r \, da(X) \quad (4.26)$$

However, we have the result

Proposition 4.7 If $\underline{S}_t \cap P$ is a plane surface, if the limit in (4.17) is valid and if

$$\begin{aligned} \text{Div}(\Phi_p \otimes \tilde{c}) &\in L^1(P), \quad \Phi_p \tilde{c} \cdot \tilde{n} \in L^1(\partial P) \quad \text{and} \\ [\Phi_p] \tilde{c} \cdot \tilde{M} &\in L^1(\underline{S}_t \cap P) \end{aligned} \quad (4.27)$$

then

$$R(P, t) = \lim_{R \rightarrow 0} \int_{\underline{U}_t(R) \cap P} K_p \tilde{e}_r \, da(X) \quad (4.28)$$

Proof: We use the technique presented in the proof of Proposition 2.9. The divergence theorem gives (see Figure 2.17)

$$\int_{P_\delta} \text{Div}(\Phi_p \otimes \tilde{c}) \, dv(X) = \int_{a_\delta} \Phi_p \tilde{c} \cdot \tilde{n} \, da(X) - \int_{b_\delta} \Phi_p \tilde{c} \, da(X) - \int_{\underline{S}_t \cap P_\delta} [\Phi_p] \tilde{c} \cdot \tilde{M} \, da(X) \quad (4.29)$$

where P_δ , a_δ and b_δ are defined in the proof of Proposition 2.9 (cf. formula (2.84)). By taking limits in (4.29) (as $\delta \rightarrow 0$) and using (4.27) we get

$$\int_P \text{Div}(\Phi_p \otimes \tilde{c}) \, dv(X) = \int_{\partial P} \Phi_p \tilde{c} \cdot \tilde{n} \, da(X) - \int_{\underline{S}_t \cap P} [\Phi_p] \tilde{c} \cdot \tilde{M} \, da(X) \quad (4.30)$$

On the other hand we have (analogous to (4.24))

$$\begin{aligned} \int_{\underline{U}_t(R) \cap P} \Phi_p \tilde{c} \cdot \tilde{e}_r \, da(X) &= \int_{\partial P \setminus \underline{A}_t^1(R)} \Phi_p \tilde{c} \cdot \tilde{n} \, da(X) - \int_{\underline{S}_t \cap \underline{P}_t(R)} [\Phi_p] \tilde{c} \cdot \tilde{M} \, da(X) - \\ &\quad \int_{\underline{P}_t(R)} \text{Div}(\Phi_p \otimes \tilde{c}) \, da(X) \end{aligned} \quad (4.31)$$

where we have used the fact that $\Phi_p \tilde{c} \cdot \tilde{m} = 0$ on $\underline{S}_t \cap P = \emptyset$. By taking limits in (4.31) and using (4.30) we get

$$\lim_{R \rightarrow 0} \int_{\underline{U}_t(R) \cap P} \Phi_p \tilde{c} \cdot \tilde{e}_r \, da(X)$$

and this proves the Proposition. $\square$

Remark: If we, in addition to the assumptions in Proposition (4.7), have

$\text{Div } K_p \in L^1(P)$ and $[K_p] \tilde{M} \in L^1(\tilde{\Sigma}_t \cap P)$ then

$$R(P, t) = \int_{\partial P} K_p \tilde{n} da(X) - \int_{\tilde{\Sigma}_t \cap P} [K_p] \tilde{m} da(X) - \int_{\tilde{\Sigma}_t \cap P} [K_p] \tilde{M} da(X) - \int_P \text{Div } K_p da(X) \quad (4.32)$$

This follows from Proposition 4.6 and (4.28).

Finally, in this section, we present a "localized" version of the reduced master balance law. Let $\mathcal{Q}_t(R, s)$ denote the cross-sectional curve of $\mathcal{T}_{u_t}(R)$ at the point s on $\mathcal{F}_t$, i.e. a circle of radius R , situated in the tubular cross-section at s and centered at the crack-front (cf. section 2.3). If $(\text{Res})_p$ is valid and if P is an interior subbody of $\mathcal{B}$ then we have (R sufficiently small)

$$\mathcal{T}_{u_t}(R) \int_P (K_p + \Phi_p \otimes \tilde{\epsilon}) \tilde{e}_r da(X) = \int_{\mathcal{F}_t \cap P} \left(\int_{\mathcal{Q}_t(R, s)} (K_p + \Phi_p \otimes \tilde{\epsilon}) \tilde{e}_r \tilde{A} ds'(X) \right) ds(X) \quad (4.33)$$

where the geometric factor $\tilde{A}$ is given by (cf. (2.69))

$$\tilde{A} = \tilde{A}_t(s, s') \equiv |1 - R \kappa_t(s) \cos(\frac{s-s'}{R})| \quad (4.34)$$

Here s and s' are the arc-length coordinates on $\mathcal{F}_t$ and $\mathcal{Q}_t(R, s)$ respectively and κ_t denotes the curvature of $\mathcal{F}_t$.

Theorem 4.2 If the limit in (4.17) is valid (or if formula (4.10) is valid) and if (for $t \in T$ fixed)

$$\int_{\mathcal{Q}_t(R, s)} (K_p + \Phi_p \otimes \tilde{\epsilon}) \tilde{e}_r \tilde{A} ds'(X) \quad (4.35)$$

converges as $R \rightarrow 0$, to a limit-function uniformly in $s \in [a, b]$ for every $a, b \in \mathbb{R}$, $s_1(t) < a < b < s_2(t)$, then the reduced master balance law (restricted to interior subbodies) is equivalent to

$$[\Psi_p \tilde{\epsilon} + k_p \tilde{\nu}] - \Pi_{FP} = \lim_{R \rightarrow 0} \int_{\mathcal{Q}_t(R, s)} (K_p + \Phi_p \otimes \tilde{\epsilon}) \tilde{e}_r \tilde{A} ds'(X) \quad (\text{Lrmb1})_p$$

at $X = \hat{X}(s, t) \in \hat{\mathcal{F}}_t$. We call this equation the local form of the reduced master balance law.

Proof: Let P be an interior subbody. From $(\text{Res})_p$, (4.33) and a standard result on uniform convergence we conclude that the reduced master balance law is equivalent to

$$\int_{F_t \cap P} (\Psi_p \tilde{c} + k_p \tilde{v}) - \pi_{Fp} - \lim_{R \rightarrow 0} \int_{\partial_t(R,s)} (K_p + \Phi_p \otimes \tilde{c}) \tilde{e}_r \tilde{A} ds'(X) ds(X) = 0$$

and this proves the Theorem. $\square$

5. BALANCE LAWS IN A THERMO-MECHANICAL THEORY OF FRACTURE

The discussion presented thus far has been very general and it possesses the potential of several concrete applications.

In this chapter we discuss a model involving the case of a non-polar, single-phase continuum. We present, in the form of a catalogue, the balance laws for mass, momentum, moment of momentum, energy and entropy for a fracturing body, by giving specific thermo-mechanical interpretations to the abstract fields $\Phi, \Pi, \dots, \Psi^\pm, \Pi^\pm, \dots$ introduced in chapter 3 and by invoking the local balance laws 1 - 5 and the reduced master balance law. The reduced master balance law is taken in its local form (Lrmb1). The reader is reminded of the fact that the local form is equivalent to the global form (Rmb1) under the regularity assumptions specified in section 4.1. In section 5.2 we introduce the concept Moderate fracture-motion. This is motivated by the conjecture, mentioned in section 2.3, that the convected crack-front velocity-field x' will be regular at the crack-front. The simplifications of the local balance at a crack-front material point, based on this assumption, will lead up to the path-independent integrals introduced by Rice and Sih. This is discussed in section 5.3.

5.1 The catalogue of balance laws

MASS: We take

$$\begin{aligned}\Phi &= \rho && \text{bulk mass density} \\ \Psi^\pm &= \rho_S^\pm && \text{crack-surface mass density}\end{aligned}$$

and all the other fields are taken equal to zero. We assume that

$$\begin{aligned}\rho(X, t) &\geq 0, & X \in \mathcal{B} \setminus \tilde{\mathcal{S}}_t \\ \rho_S^\pm(X, t) &\geq 0, & X \in \tilde{\mathcal{S}}_t \setminus \tilde{\mathcal{F}}_t\end{aligned}\tag{5.1}$$

The balance laws are (in the referential description)

$$\dot{\rho}_p = 0 \tag{Lb1 1}_p$$

$$[\rho_p]_{\tilde{\omega}} = 0 \tag{Lb1 2}_p$$

$$\dot{\rho}_{Sp}^{\pm} = 0 \quad (\text{Lb13})_p$$

$$\int_{\rho_{Sp}} [\zeta] = \lim_{R \rightarrow 0} \int_{\partial_t^-(R,s)} \rho_p \tilde{\zeta} \cdot \tilde{e}_r \tilde{\zeta} ds'(X) \quad (\text{Lrmb1})_p$$

The local balance laws 4 and 5 are satisfied trivially. Here ρ_p and $\rho_{Sp}^{\pm}$ are the Piola-fields corresponding to ρ and $\rho_S^{\pm}$ respectively. It is evident that $(\text{Lb11})_p$ and $(\text{Lb13})_p$ are satisfied if and only if

$$\begin{aligned} \rho_p &= \rho_p(X), & X &\in B \setminus \tilde{S}_t \\ \rho_{Sp}^{\pm} &= \rho_{Sp}^{\pm}(X), & X &\in \tilde{S}_t \setminus \tilde{F}_t \end{aligned} \quad (5.2)$$

Proposition 5.1 $(\text{Lrmb1})_p$ is satisfied if and only if

$$\rho_{Sp}^{\pm}(X) = 0, \quad X \in \tilde{S}_t \setminus \tilde{F}_t \quad (5.3)$$

Proof: Let $Z \in \tilde{S}_t$. Then, for some $t^- < t$, we have $Z \in \tilde{F}_{t^-}$, $\zeta_{t^-}(Z) > 0$ and $(\text{Lrmb1})_p$ (at time t^-) is equivalent to

$$\int_{\rho_{Sp}} [Z] = \lim_{R \rightarrow 0} \int_{\partial_t^-(R,s)} \rho_p \tilde{\zeta} \cdot \tilde{e}_r \tilde{\zeta} ds'(X), \quad Z = \hat{X}(s, t^-) \quad (5.4)$$

Since ρ_p must be bounded in a neighbourhood of Z we have the simple estimate

$$\left| \int_{\partial_t^-(R,s)} \rho_p \tilde{\zeta} \cdot \tilde{e}_r \tilde{\zeta} ds'(X) \right| \leq C(1 + R \zeta_{t^-}(s)) R \quad (5.5)$$

where C is a positive constant. From (5.4) and (5.5) we conclude that $\int_{\rho_{Sp}} [Z] = 0$ and consequently $\rho_{Sp}^{\pm}(Z) = 0$.

Conversely if $\rho_{Sp}^{\pm}(X) = 0$, $X \in \tilde{S}_t \setminus \tilde{F}_t$ then $\int_{\rho_{Sp}} [X] = 0$, $X \in \tilde{F}_t$. Now if $X \in \tilde{F}_t$ and $\zeta_t(X) = 0$ then $(\text{Lrmb1})_p$ is satisfied trivially. If $\zeta_t(X) > 0$ then, from (5.5), we may conclude that the limit in $(\text{Lrmb1})_p$ is equal to zero. This proves the Proposition. $\square$

Remark : A direct application of the (global) master balance law (Mb1) gives (since $P(P, t) = S(P, t) = 0$)

$$0 = \frac{d}{dt} \left(\int_P \rho_p dv(X) + \int_{\tilde{S}_t \cap P} (\rho_{Sp}^+ + \rho_{Sp}^-) da(X) \right) = \int_{\tilde{F}_t \cap P} \int_{\rho_{Sp}} [\zeta] ds(X) \quad (5.6)$$

where we have used $(Lb11)_p$, $(Lb13)_p$ and the surface transport theorem. The relation (5.6) may now be used to give another (and more direct) proof of Proposition 5.1. ' .

If $\rho(X,t) > 0$, $X \in \mathcal{B} \setminus \Sigma_t$ then we may introduce the mass-specific fracture-fields φ , π and q defined by

$$\varphi \equiv \frac{1}{\rho} \Phi = \frac{1}{\rho_p} \Phi_p, \quad \pi \equiv \frac{1}{\rho} \Pi = \frac{1}{\rho_p} \Pi_p, \quad q \equiv \frac{1}{\rho} Q = \frac{1}{\rho_p} Q_p \quad (5.7)$$

The local balance laws 1 and 2 may then be written (if mass is balanced)

$$(\dot{\varphi} - \pi - q) \rho - \operatorname{div} K = 0 \quad (Lb11)$$

$$(\dot{\varphi} - \pi - q) \rho_p - \operatorname{Div} K_p = 0 \quad (Lb11)_p$$

$$[\varphi] \mu + [K] \tilde{M} + \pi_{\Sigma} = 0 \quad (Lb12)$$

$$[\varphi] \mu_p + [K_p] \tilde{M}_p + \pi_{\Sigma p} = 0 \quad (Lb12)_p$$

where

$$\mu \equiv \rho \omega, \quad \mu_p \equiv A_{\Sigma} \mu = \rho_p \tilde{\omega} \quad (5.8)$$

and A_{Σ} is the area-ratio associated with the restriction of χ to Σ .

MOMENTUM: We take

$$\varphi = \dot{\mathbf{x}} \quad \text{material velocity}$$

$$q = \bar{\mathbf{b}} \quad \text{specific external body force}$$

$$K = \underline{\underline{\mathbf{T}}} \quad \text{Cauchy stress tensor}$$

$$R^{\pm} = \bar{\mathbf{t}}_e^{\pm} \quad \text{"external" force on the crack-surfaces}$$

$$k^{\pm} = \hat{\Sigma}^{\pm} \quad \text{surface stress tensor}$$

$$R^{\partial} = \bar{\mathbf{t}}_e^{\partial} \quad \text{external force on the boundary}$$

and all the remaining fields are taken equal to zero. We assume that

$$\hat{\Sigma}^{\pm}(X,t) \in L(T_{X^{\pm}}^{\pm}(X,t), V).$$

The balance laws are (in the referential description)

$$(\ddot{\mathbf{x}} - \bar{\mathbf{b}}) \rho_p - \text{Div } \mathbf{T}_p = \bar{\mathbf{0}} \quad (\text{Lb1 1})_p$$

$$[\dot{\mathbf{x}}] \mu_p + [\mathbf{T}_p] \bar{\mathbf{m}} = \bar{\mathbf{0}} \quad (\text{Lb1 2})_p$$

$$- \bar{\mathbf{t}}_{ep}^{\pm} + (\mathbf{T}_p)^{\pm} \bar{\mathbf{m}} - \text{Div}_s \hat{\mathbf{S}}_p^{\pm} = \bar{\mathbf{0}} \quad (\text{Lb1 3})_p$$

$$- \bar{\mathbf{t}}_{ep}^{\partial} + \mathbf{T}_p \bar{\mathbf{n}} = \bar{\mathbf{0}} \quad (\text{Lb1 4})_p$$

$$\hat{\mathbf{S}}_p^{\pm} \bar{\mathbf{v}} = \bar{\mathbf{0}} \quad (\text{Lb1 5})_p$$

$$\hat{\mathbf{J}}_p[\bar{\mathbf{v}}] = \lim_{R \rightarrow 0} \int_{\partial_t^{\pm}(R, s)} (\mathbf{T}_p + \dot{\mathbf{x}} \rho_p \otimes \bar{\mathbf{c}}) \bar{\mathbf{e}}_r \bar{\mathbf{A}} ds'(X) \quad (\text{Lrmb1})_p$$

Remark: Let $p_t^{\pm}$ be a regular region on $S_t^{\pm}$ with outward pointing unit normal vector $\bar{\mathbf{v}}_t^{\pm}(x) \in T_x^{\pm}$, $x \in \partial p_t^{\pm}$. The vector

$$\hat{\mathbf{S}}^{\pm}(x, t) \bar{\mathbf{v}}_t^{\pm}(x)$$

represents the force per unit length (in the spatial description) at $x \in \partial p_t^{\pm}$ exerted by $S_t^{\pm} \setminus p_t^{\pm}$ on $p_t^{\pm}$.

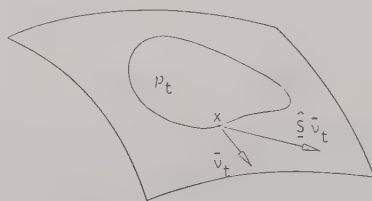


Figure 5.1

The external force $\bar{\mathbf{t}}_e^{\pm}$ represents the force per-unit area (in the spatial description) exerted on $S_t^{\pm}$ by the environment of B or interactions (contact forces) between the crack-surfaces.

The external force $\bar{\mathbf{t}}_e^{\partial}$ represents the force per unit area exerted on $\partial B_t \setminus (S_t^+ \cup S_t^-)$ by the environment of B .

$\mathbf{T}_p$ is called the first Piola-Kirchhoff (or nominal) bulk stress tensor, and $\hat{\mathbf{S}}_p$ is called the first Piola-Kirchhoff surface stress tensor.

MOMENT OF MOMENTUM: We take

$$\varphi = (x - x_0) \times \dot{x} \quad R^{\pm} = (x - x_0) \times \tilde{t}_e^{\pm}$$

$$q = (x - x_0) \times \tilde{b} \quad k^{\pm} = (x - x_0) \times \hat{S}_{\pm}^{\pm}$$

$$K = (x - x_0) \times \tilde{I} \quad R^{\partial} = (x - x_0) \times \tilde{t}_e^{\partial}$$

where x_0 is an arbitrary fix point in E . All the remaining fields are taken equal to zero. The balance laws are (in the referential description)

$$(x - x_0) \times (\ddot{x} - \tilde{b}) \rho_p - \text{Div}((x - x_0) \times \tilde{I}_p) = \tilde{0} \quad (\text{Lb11})_p$$

$$(x - x_0) \times ([\dot{x}] \mu_p + [\tilde{I}_p] \tilde{M}) = \tilde{0} \quad (\text{Lb12})_p$$

$$(x^{\pm} - x_0) \times (-\tilde{t}_{ep}^{\pm} \mp (\tilde{I}_p)^{\pm} \tilde{M}) - \text{Div}_{\mathcal{A}}((x^{\pm} - x_0) \times \hat{S}_p^{\pm}) = \tilde{0} \quad (\text{Lb13})_p$$

$$(x - x_0) \times (-\tilde{t}_{ep}^{\partial} + \tilde{I}_p \tilde{n}) = \tilde{0} \quad (\text{Lb14})_p$$

$$(x^{\pm} - x_0) \times \hat{S}_p^{\pm} \tilde{\gamma} = \tilde{0} \quad (\text{Lb15})_p$$

$$(x^{\pm} - x_0) \times \hat{S}_p^{\pm} [\tilde{\gamma}] = \lim_{R \rightarrow 0} \int_{\mathcal{Q}_t(R,s)} (x - x_0) \times (\tilde{I}_p + \dot{x} \rho_p \otimes \tilde{\gamma}) \tilde{e}_r \tilde{A} ds'(X) \quad (\text{Lrmb1})_p$$

Proposition 5.2 If mass and momentum are balanced quantities then

- $(\text{Lb12})_p$, $(\text{Lb14})_p$ and $(\text{Lb15})_p$ are all satisfied trivially.
- $(\text{Lb11})_p$ is equivalent to the condition $\tilde{I}_p^T = \tilde{I}_p$
- $\hat{S}_p^{\pm}(X,t) \tilde{\gamma} \in \mathcal{T}_X^{\pm}(X,t)$ for all $\tilde{\gamma} \in \mathcal{T}_X$ ($X \in \mathcal{S}_t$) and $(\text{Lb13})_p$ is equivalent to the condition

$$\tilde{I}_p^{\pm} (\hat{S}_p^{\pm})^T \tilde{I}_p^{\pm} = \tilde{I}_p^{\pm} \hat{S}_p^{\pm} (\tilde{I}_p^{\pm})^T \quad (5.9)$$

- $(\text{Lrmb1})_p$ is equivalent to the condition

$$\tilde{0} = \lim_{R \rightarrow 0} \int_{\mathcal{Q}_t(R,s)} (x - x^{\pm}(\hat{X}(s,t),t)) \times (\tilde{I}_p + \dot{x} \rho_p \otimes \tilde{\gamma}) \tilde{e}_r \tilde{A} ds'(X) \quad (5.10)$$

Proof: a) is evident. b) follows from the relation $\text{Div}((\chi - x_0) \times \underline{T}_p) = (\chi - x_0) \times \text{Div} \underline{T}_p + \alpha \chi (\underline{F}_p^T - \underline{T}_p \underline{F}^T)$ and the local balance of momentum at a bulk material point. c) We have

$$\text{Div}_\Delta((\chi^\pm - x_0) \times \hat{\underline{S}}_p^\pm)^1 = (\chi^\pm - x_0) \times \text{Div}_\Delta \hat{\underline{S}}_p^\pm + \alpha \chi (\bar{\underline{F}}^\pm (\hat{\underline{S}}_p^\pm)^T - \hat{\underline{S}}_p^\pm (\bar{\underline{F}}^\pm)^T) \quad (5.11)$$

From (5.11) and the local balance of momentum at a crack-surface material point it now follows that $(\text{Lb13})_p$ is equivalent to

$$\bar{\underline{F}}^\pm (\hat{\underline{S}}_p^\pm)^T = \hat{\underline{S}}_p^\pm (\bar{\underline{F}}^\pm)^T \quad (5.12)$$

Let $\bar{\underline{\nu}} \in T_X$, $X \in \underline{\mathcal{S}}_t$ and let $\bar{\underline{m}}^\pm = \bar{\underline{m}}_t^\pm(\chi^\pm(X, t))$ be the unit normal on $S_t^\pm$, then

$$\begin{aligned} \bar{\underline{m}}^\pm \cdot \hat{\underline{S}}_p^\pm \bar{\underline{\nu}} &= \bar{\underline{m}}^\pm \cdot \hat{\underline{S}}_p^\pm (\bar{\underline{F}}^\pm)^T (\bar{\underline{F}}^\pm)^{-T} \bar{\underline{\nu}} = \bar{\underline{F}}^\pm (\hat{\underline{S}}_p^\pm)^T \bar{\underline{m}}^\pm \cdot (\bar{\underline{F}}^\pm)^{-T} \bar{\underline{\nu}} = \\ \hat{\underline{S}}_p^\pm (\bar{\underline{F}}^\pm)^T \bar{\underline{m}}^\pm \cdot (\bar{\underline{F}}^\pm)^{-T} \bar{\underline{\nu}} &= \hat{\underline{S}}_p^\pm (\bar{\underline{F}}^\pm)^T \underline{P}^\pm \bar{\underline{m}}^\pm \cdot (\bar{\underline{F}}^\pm)^{-T} \bar{\underline{\nu}} = 0 \end{aligned} \quad (5.13)$$

where we have used (2.6) and (5.12). It is now a simple matter to confirm that (5.12) is equivalent to (5.9). d) We have

$$\begin{aligned} \int_{\mathcal{Q}_t(R, s)} (\chi - \chi^\pm(\hat{\chi}(s, t), t)) \times (\underline{T}_p + \dot{\chi} \rho_p \otimes \bar{\underline{\zeta}}) \bar{\underline{e}}_r \underline{A} ds'(X) = \\ \int_{\mathcal{Q}_t(R, s)} (\chi - x_0) \times (\underline{T}_p + \dot{\chi} \rho_p \otimes \bar{\underline{\zeta}}) \bar{\underline{e}}_r \underline{A} ds'(X) - \\ (\chi^\pm(\hat{\chi}(s, t), t) - x_0) \times \int_{\mathcal{Q}_t(R, s)} (\underline{T}_p + \dot{\chi} \rho_p \otimes \bar{\underline{\zeta}}) \bar{\underline{e}}_r \underline{A} ds'(X) \end{aligned} \quad (5.14)$$

By taking limits in (5.14) (as $R \rightarrow 0$) and using the local balance of momentum at a crack-front material point we infer that $(\text{Lrmb1})_p$ is equivalent to (5.10). $\square$

Remark 1: Take $\underline{S}^\pm = \underline{P}^\pm \hat{\underline{S}}^\pm$ then, according to c), we have

$$\begin{aligned} \underline{S}^\pm(X) \bar{\underline{\nu}}^\pm &= \hat{\underline{S}}^\pm(X) \bar{\underline{\nu}}^\pm \text{ for every } \bar{\underline{\nu}}^\pm \in T_X^\pm \\ \underline{S}^\pm(X) &\in \text{Sym}(T_X^\pm) \end{aligned} \quad (5.15)$$

Note that $\underline{S}_p^\pm = \hat{\underline{S}}_p^\pm$.

Remark 2: For an extensive discussion on surface stress see Gurtin & Murdoch [13].

Remark 3: A generalized model for the crack-surface is obtained if we take

$$R^{\pm} = (x - x_0) \times \bar{t}_e^{\pm} + \bar{l}^{\pm}, \quad k^{\pm} = (x - x_0) \times \hat{\underline{S}}^{\pm} + \underline{M}^{\pm}$$

where $\bar{l}^{\pm}(X, t) \in V$ is the external couple and $\underline{M}^{\pm}(X, t) \in L(\tau_X^{\pm}(X, t), V)$ is the surface couple-stress tensor. The local balance of moment of momentum at a crack-surface material point is then equivalent to

$$- \bar{l}_p^{\pm} - \alpha x (\bar{\underline{E}}^{\pm} (\hat{\underline{S}}_p^{\pm})^T - \hat{\underline{S}}_p^{\pm} (\bar{\underline{E}}^{\pm})^T) - \text{Div}_{\Delta} \underline{M}_p^{\pm} = \bar{0}$$

Remark 4: The condition in b) is of course equivalent to $\underline{\underline{I}}^T = \underline{\underline{I}}$.

ENERGY: We take

$$\begin{aligned} \varphi &= \epsilon + \frac{1}{2} \dot{x} \cdot \dot{x} & \epsilon &: \text{specific (mass-) internal bulk energy} \\ q &= s + \bar{b} \cdot \dot{x} & s &: \text{specific external heat supply to the bulk} \\ K &= \bar{h} + \underline{\underline{I}}^T \dot{x} & \bar{h} &: \text{bulk heat flux vector} \\ \Psi^{\pm} &= \epsilon_S^{\pm} & &: \text{internal surface energy} \\ R^{\pm} &= R_{\epsilon}^{\pm} & &: \text{"external" energy supply to the crack-surfaces} \\ k^{\pm} &= \bar{h}_S^{\pm} + (\underline{\underline{S}}^{\pm})^T \dot{x}_{||} & \bar{h}_S^{\pm} &: \text{crack-surface heat flux vector} \\ R^{\partial} &= R_{\epsilon}^{\partial} & &: \text{external energy supply to the boundary} \end{aligned}$$

and all the remaining fields are taken equal to zero. We assume that $\bar{h}_S^{\pm}(X, t) \in \tau_X^{\pm}(X, t)$. The balance laws are (in the referential description)

$$(\dot{\epsilon} + (\ddot{x} - \bar{b}) \cdot \dot{x} - s) \rho_p - \text{Div}(\bar{h}_p + \underline{\underline{I}}_p^T \dot{x}) = 0 \quad (\text{Lb1 } 1)_p$$

$$[\epsilon + \frac{1}{2} \dot{x} \cdot \dot{x}] \nu_p + [\bar{h}_p + \underline{\underline{I}}_p^T \dot{x}] \cdot \bar{m} = 0 \quad (\text{Lb1 } 2)_p$$

$$\dot{\epsilon}_{Sp}^{\pm} - R_{\epsilon p}^{\pm} \mp (\bar{h}_p + \underline{\underline{I}}_p^T \dot{x})^{\pm} \cdot \bar{m} - \text{Div}_{\Delta} (\bar{h}_{Sp}^{\pm} + (\underline{\underline{S}}_p^{\pm})^T \dot{x}^{\pm}) = 0 \quad (\text{Lb1 } 3)_p$$

$$- R_{\epsilon p}^{\partial} + (\bar{h}_p + \underline{\underline{I}}_p^T \dot{x}) \cdot \bar{m} = 0 \quad (\text{Lb1 } 4)_p$$

$$(\bar{h}_{Sp}^{\pm} + (S_p^{\pm})^T \dot{x}^{\pm}) \cdot \bar{\nu} = 0 \quad (Lb15)_p$$

$$] \epsilon_{Sp} [\bar{c} +] \bar{h}_{Sp} + S_p^T \dot{x} [\cdot \bar{\nu} = \lim_{R \rightarrow 0} \int_{\partial_t(R,s)} (\bar{h}_p + S_p^T \dot{x} + (\epsilon + \frac{1}{2} \dot{x} \cdot \dot{x}) \rho_p \bar{c}) \cdot \bar{e}_r A ds'(X) \quad (Lrmb1)_p$$

Proposition 5.3 If mass, momentum and moment of momentum are balanced quantities then

a) $(Lb11)_p$ is equivalent to the condition

$$(\dot{\epsilon} - s) \rho_p - I_p \cdot \dot{\bar{F}} - \text{Div} \bar{h}_p = 0 \quad (5.16)$$

b) $(Lb13)_p$ is equivalent to the condition

$$\dot{\epsilon}_{Sp}^{\pm} - R_{\epsilon p}^{\pm} + \bar{t}_{ep}^{\pm} \cdot \dot{x}^{\pm} \mp (\bar{h}_p)^{\pm} \bar{m} - S_p^{\pm} \cdot \dot{\bar{F}}^{\pm} - \text{Div}_{\Delta} \bar{h}_{Sp}^{\pm} = 0 \quad (5.17)$$

c) $(Lb14)_p$ is equivalent to the condition

$$- R_{\epsilon p}^{\partial} + \bar{t}_{ep}^{\partial} \cdot \dot{x} + \bar{h}_p \cdot \bar{n} = 0 \quad (5.18)$$

d) $(Lb15)_p$ is equivalent to the condition

$$\bar{h}_{Sp}^{\pm} \cdot \bar{\nu} = 0 \quad (5.19)$$

Proof: a) and b) follow from the relations

$$\text{Div}(S_p^T \dot{x}) = (\text{Div} I_p) \cdot \dot{x} + I_p \cdot \dot{\bar{F}}, \quad \text{Div}_{\Delta}((S_p^{\pm})^T \dot{x}^{\pm}) = (\text{Div}_{\Delta} S_p^{\pm}) \cdot \dot{x}^{\pm} + S_p^{\pm} \cdot \dot{\bar{F}}^{\pm}$$

and the local balance of momentum. c) and d) are direct consequences of the local balance of momentum. $\square$

Remark: The external energy supply $R_{\epsilon}^{\pm}$ represents energy supply to the crack-surfaces from the environment of B or energy-interactions between the crack-surfaces.

The external energy supply R_{ϵ}^{∂} represents energy supply to the boundary $\partial B_t \setminus (S_t^+ \cup S_t^-)$ from the environment of B .

ENTROPY: We take

$\varphi = \eta$	specific bulk entropy
$\pi = \gamma$	specific bulk entropy production
$q = \frac{1}{\theta} s$	θ : bulk temperature
$K = \frac{1}{\theta} \bar{h}$	
$\psi^\pm = \eta_S^\pm$	crack-surface entropy
$\pi_S^\pm = \gamma_S^\pm$	crack-surface entropy production
$R^\pm = R_\eta^\pm$	"external" entropy supply to the crack-surfaces
$k^\pm = \frac{1}{\theta_S^\pm} \bar{h}_S^\pm$	$\theta_S^\pm$: crack-surface temperature
$\pi_\Sigma = \gamma_\Sigma$	singular surface entropy production
$\pi_F = \gamma_F$	crack-front entropy production
$\pi^\partial = \gamma^\partial$	boundary entropy production
$R^\partial = R_\eta^\partial$	external entropy supply to the boundary

We assume that

$$\begin{aligned} \theta(X, t) &> 0, & X \in B \setminus \underline{S}_t \\ \theta_S^\pm(X, t) &> 0, & X \in \underline{S}_t \setminus \underline{F}_t \end{aligned} \quad (5.20)$$

The balance laws are (in the referential description)

$$(\dot{\eta} - \gamma - \frac{1}{\theta} s) \rho_p - \text{Div}(\frac{1}{\theta} \bar{h}_p) = 0 \quad (\text{Lb1 1})_p$$

$$[\eta] \mu_p + [\frac{1}{\theta} \bar{h}_p] \cdot \bar{M} + \gamma_{\Sigma p} = 0 \quad (\text{Lb1 2})_p$$

$$\dot{\eta}_{Sp}^\pm - \gamma_{Sp}^\pm - R_{\eta p}^\pm \mp (\frac{1}{\theta} \bar{h}_p)^\pm \cdot \bar{M} - \text{Div}_\Delta(\frac{1}{\theta_S^\pm} \bar{h}_{Sp}^\pm) = 0 \quad (\text{Lb1 3})_p$$

$$-\gamma_p^\partial - R_{\eta p}^\partial + \frac{1}{\theta} \bar{h}_p \cdot \bar{N} = 0 \quad (\text{Lb1 4})_p$$

$$\frac{1}{\theta_S^+} \bar{h}_{Sp}^+ \cdot \bar{\gamma} = 0 \quad (\text{Lb1 5})_p$$

$$J_{Sp}[\bar{\gamma} + \frac{1}{\theta_S} \bar{h}_{Sp}[\bar{\gamma} - \gamma_{Fp}] = \lim_{R \rightarrow 0} \int_{\bar{\Sigma}_t(R,s)} (\frac{1}{\theta} \bar{h}_p + \eta \rho_p \bar{\gamma}) \cdot \bar{\Sigma}_r \bar{\Delta} ds'(X) \quad (\text{Lrmb1})_p$$

We introduce

$$\psi \equiv \varepsilon - \theta \eta \quad \text{specific free bulk energy}$$

$$\bar{g} \equiv \text{Grad } \theta \quad \text{referential bulk temperature gradient}$$

$$\psi_S^\pm \equiv \varepsilon_S^\pm - \theta_S^\pm \eta_S^\pm \quad \text{crack-surface free energy}$$

$$\bar{g}_S^\pm \equiv \text{Grad}_\Delta \theta_S^\pm \quad \text{referential crack-surface temperature gradient}$$

Proposition 5.4 If mass, momentum, moment of momentum and energy are balanced quantities then

a) (Lb1 1)_p is equivalent to the condition

$$(\dot{\psi} + \dot{\theta} \eta + \theta \gamma) \rho_p - \bar{T}_p \cdot \dot{\bar{F}} - \frac{1}{\theta} \bar{h}_p \cdot \bar{g} = 0 \quad (5.21)$$

b) (Lb1 2)_p is equivalent to the condition

$$[\psi + \theta \{n\}] \mu_p + [\dot{\chi}] \cdot \{\bar{T}_p\} \bar{M} + [(1 - \frac{\{\theta\}}{\theta}) \bar{h}_p] \cdot \bar{M} - \{\theta\} \gamma_{zp} = 0 \quad (5.22)$$

where the mean value of $\eta(\cdot, t)$ on $\bar{\Sigma}_t$ is denoted $\{n\}(\cdot, t)$ and defined by

$$\{n\}(X, t) \equiv \frac{1}{2}((n)^+(X, t) + (n)^-(X, t)), \quad X \in \bar{\Sigma}_t \setminus \Sigma_t \quad (5.23)$$

c) (Lb1 3)_p is equivalent to the condition

$$\begin{aligned} \dot{\psi}_{Sp}^\pm + \dot{\theta}_S^\pm \eta_{Sp}^\pm + \theta_S^\pm \gamma_{Sp}^\pm - \bar{S}_p^\pm \cdot \dot{\bar{F}}^\pm - \frac{1}{\theta_S^\pm} \bar{h}_{Sp}^\pm \cdot \bar{g}_S^\pm + \theta_S^\pm R_{\eta p}^\pm - R_{\varepsilon p}^\pm + \\ \bar{t}_{ep}^\pm \cdot \dot{\chi}^\pm \mp (1 - \frac{\theta_S^\pm}{(\theta)^\pm}) (\bar{h}_p)^\pm \cdot \bar{m} = 0 \end{aligned} \quad (5.24)$$

d) (Lb1 4)_p is equivalent to the condition

$$-\theta \gamma_p^\partial - \theta R_{\eta p}^\partial + R_{\varepsilon p}^\partial - \bar{t}_{ep}^\partial \cdot \dot{\chi} = 0 \quad (5.25)$$

- e) $(Lb15)_p$ is satisfied trivially.
 f) $(Lrmb1)_p$ is equivalent to the condition

$$\begin{aligned} &]\psi_{Sp}[\tilde{\omega} +]\Sigma_p^T \dot{\tilde{\omega}} \cdot \tilde{\omega} + \frac{1}{2} [\theta_S] ([\eta_{Sp}] \tilde{\omega} + [\frac{1}{\theta_S} \bar{h}_{Sp}] \cdot \tilde{\omega}) + \{\theta_S\} \gamma_{Fp} = \\ & \lim_{R \rightarrow 0} \int_{\tilde{\omega}_t(R,s)} \left(\left(1 - \frac{\{\theta_S\}}{\theta} \right) \bar{h}_p + \Sigma_p^T \dot{\tilde{\omega}} + \right. \\ & \left. (\psi + \frac{1}{2} \dot{\tilde{\omega}} \cdot \dot{\tilde{\omega}} + (1 - \frac{\{\theta_S\}}{\theta} \eta) \rho_p \tilde{\omega}) \cdot \tilde{e}_r \tilde{\omega} ds'(X) \right) \end{aligned} \quad (5.26)$$

where the jump $[\theta_S]$ and the mean value $\{\theta_S\}$ are defined by

$$\begin{aligned} [\theta_S](X,t) &= \theta_S^+(X,t) - \theta_S^-(X,t), \\ \{\theta_S\}(X,t) &= \frac{1}{2} (\theta_S^+(X,t) + \theta_S^-(X,t)), \quad X \in \tilde{F}_t \end{aligned} \quad (5.27)$$

Proof: a) follows from (5.16) and the relation

$$\text{Div}(\frac{1}{\theta} \bar{h}_p) = \frac{1}{\theta} \text{Div} \bar{h}_p - \frac{1}{\theta^2} \bar{h}_p \cdot \tilde{g} \quad (5.28)$$

b) follows from the local balance of energy and the simple rule

$$[\varphi_1 \varphi_2] = [\varphi_1] \{\varphi_2\} + \{\varphi_1\} [\varphi_2] \quad (5.29)$$

c) follows from (5.17) and the relation

$$\text{Div}_S(\frac{1}{\theta_S^+} \bar{h}_{Sp}^+) = \frac{1}{\theta_S^+} \text{Div}_S \bar{h}_{Sp}^+ - \frac{1}{(\theta_S^+)^2} \bar{h}_{Sp}^+ \cdot \tilde{g}_S^+ \quad (5.30)$$

d) follows from (5.18) and e) follows from (5.19). f) follows from the the local balance of energy at a crack-front material point and the rule

$$]\Psi_1 \Psi_2[= \frac{1}{2} [\Psi_1] [\Psi_2] + 2 \{ \Psi_1 \} \{ \Psi_2 \} = \frac{1}{2} [\Psi_1] [\Psi_2] + \{ \Psi_1 \}] \Psi_2 [\quad (5.31)$$

Note that $]\Psi[= 2 \{ \Psi \}$. $\square$

Corollary 5.1 If

$$\theta_S^+(X,t) = \theta_S^-(X,t), \quad X \in \tilde{F}_t$$

and if we introduce the crack-front temperature θ_F defined by

$$\theta_F(X, t) \equiv \theta_S^\pm(X, t), \quad X \in \tilde{F}_t \quad (5.32)$$

then $(Lrmb1)_p$ is equivalent to

$$\begin{aligned} J\psi_{Sp}[\tilde{\zeta}] + JS_p^T \dot{\tilde{\chi}} \cdot \tilde{\gamma} + \theta_F \gamma_{Fp} &= \lim_{R \rightarrow 0} \int_{\tilde{Q}_t(R, s)} \left(\left(1 - \frac{\theta_F}{\theta}\right) \bar{h}_p + \mathbb{I}_p^T \dot{\tilde{\chi}} + \right. \\ &\left. \left(\psi + \frac{1}{2} \dot{\tilde{\chi}} \cdot \dot{\tilde{\chi}} + \left(1 - \frac{\theta_F}{\theta}\right) \theta \eta \right) \rho_p \tilde{\zeta} \right) \cdot \tilde{e}_r \tilde{\Lambda} ds'(X) \end{aligned} \quad (5.33)$$

Proof: We have $[\theta_S] = 0$ and $\{\theta_S\} = \theta_F$. Inserting this into (5.26) proves the Corollary. $\square$

Corollary 5.2¹ If

- a) $\lim_{\substack{X \rightarrow Z \\ X \in B \setminus \tilde{S}_t}} \theta(X, t) = \lim_{\substack{X \rightarrow Z \\ X \in \tilde{S}_t \setminus \tilde{F}_t}} (\theta)^\pm(X, t) = \{\theta_S\}(Z, t), \quad Z \in \tilde{F}_t$
- b) $\int_{\tilde{Q}_t(R, s)} |\bar{h}_p| ds'(X)$ and $\int_{\tilde{Q}_t(R, s)} |\eta| \rho_p ds'(X)$ remain bounded as $R \rightarrow 0$,

then $(Lrmb1)_p$ is equivalent to

$$\begin{aligned} J\psi_{Sp}[\tilde{\zeta}] + JS_p^T \dot{\tilde{\chi}} \cdot \tilde{\gamma} + \frac{1}{2} [\theta_S] ([\eta_{Sp}] \tilde{\zeta} + [\frac{1}{\theta_S} \bar{h}_{Sp}] \cdot \tilde{\gamma}) + \{\theta_S\} \gamma_{Fp} &= \\ \lim_{R \rightarrow 0} \int_{\tilde{Q}_t(R, s)} \left(\mathbb{I}_p^T \dot{\tilde{\chi}} + \left(\psi + \frac{1}{2} \dot{\tilde{\chi}} \cdot \dot{\tilde{\chi}} \right) \rho_p \tilde{\zeta} \right) \cdot \tilde{e}_r \tilde{\Lambda} ds'(X) \end{aligned} \quad (5.34)$$

Proof: The result is an immediate consequence of the simple estimates

$$\begin{aligned} \left| \int_{\tilde{Q}_t(R, s)} \left(1 - \frac{\{\theta_S\}}{\theta}\right) \bar{h}_p \cdot \tilde{e}_r \tilde{\Lambda} ds'(X) \right| &\leq \\ (1 + R \kappa_t(s)) \int_{\tilde{Q}_t(R, s)} |\bar{h}_p| ds'(X) \sup_{X \in \tilde{Q}_t(R, s)} \left| 1 - \frac{\{\theta_S\}}{\theta(X, t)} \right| \end{aligned} \quad (5.35)$$

$$\left| \int_{\tilde{Q}_t(R, s)} \left(1 - \frac{\{\theta_S\}}{\theta}\right) \theta \eta \rho_p \tilde{\zeta} \cdot \tilde{e}_r \tilde{\Lambda} ds'(X) \right| \leq$$

¹ cf. Gurtin [12]

$$\zeta_t(s)(1 + R_{\zeta_t}(s)) \int_{Q_t(R,s)} |\eta| \rho_p ds'(X) \sup_{X \in Q_t(R,s)} |\theta(X,t) - \{\theta_S\}| \quad (5.36)$$

□

We say that we have homothermal conditions at the crack-front if the conditions a) and b) in Corollary 5.2 are satisfied and if $[\theta_S] = 0$ on the crack-front. At homo-thermal conditions the local balance of entropy at a crack-front material point is equivalent to

$$\begin{aligned} &]\psi_{Sp}[\zeta +]S_p^T \dot{\zeta} \cdot \bar{\zeta} + \theta_F \gamma_F = \\ & \lim_{R \rightarrow 0} \int_{Q_t(R,s)} (\mathbb{I}_p^T \dot{\zeta} + (\psi + \frac{1}{2} \dot{\zeta} \cdot \dot{\zeta}) \rho_p \bar{\zeta}) \cdot \bar{e}_r \bar{\zeta} ds'(X) \end{aligned} \quad (5.37)$$

We say that we have global homothermal conditions in B if θ and ζ_S are independent of X , $\bar{h} = \bar{0}$, $\bar{h}_S^\pm = \bar{0}$ and we have homothermal conditions at the crack-front.

We close this section with a few remarks.

Remark 1: The energy release rate (per unit length of the crack-front in the referential description)

$$\lim_{R \rightarrow 0} \int_{Q_t(R,s)} (\bar{h}_p + \mathbb{I}_p^T \dot{\zeta} + (\epsilon + \frac{1}{2} \dot{\zeta} \cdot \dot{\zeta}) \rho_p \bar{\zeta}) \cdot \bar{e}_r \bar{\zeta} ds'(X)$$

has been extensively discussed in the literature. See for example Atkinson & Eshelby [10], Sih [11], Freund [14], Gurtin [12,15] and Strifors [1,2]. In most cases the analysis has been carried out under more special assumptions than we have here (plane problems, no heat conduction, so called quasi-static conditions etc.).

Remark 2: It may be shown that the local balance laws at a crack-front material point are frame-indifferent. On the other hand the local balance of mass, momentum and moment of momentum at a crack-front material point may be derived by postulating the local balance of energy (at a crack-front material point) and its form-invariance under a change of frame. This technique has been used in Strifors [1,2].

Remark 3: As we have already pointed out the crack-surface fields

$$\bar{t}_e^\pm, R_e^\pm \text{ and } R_n^\pm$$

and the boundary-fields

$$\bar{t}_e^\partial, R_\epsilon^\partial, \text{ and } R_\eta^\partial$$

are introduced in order to represent the influence of the environment ("the external world") on the boundary of B (or the interactions between the crack-surfaces). If the environment consists of an adjacent exterior continuum B_e then we put

$$\begin{aligned}\bar{t}_e^\pm &\equiv (\bar{I}_e)^\pm \cdot \bar{m}^\pm, & \bar{t}_e^\partial &\equiv \bar{I}_e \cdot \bar{n} \\ R_\epsilon^\pm &\equiv (\bar{h}_e + \bar{I}_e^T \dot{x}_e)^\pm \cdot \bar{m}^\pm + r^\pm, & R_\epsilon^\partial &\equiv (\bar{h}_e + \bar{I}_e^T \dot{x}_e) \cdot \bar{n} \\ R_\eta^\pm &\equiv \left(\frac{1}{\theta_e} \bar{h}_e\right)^\pm \cdot \bar{m}^\pm + \frac{1}{\theta_S^\pm} r^\pm, & R_\eta^\partial &\equiv \left(\frac{1}{\theta_e} \bar{h}_e\right) \cdot \bar{n}\end{aligned}$$

where $\bar{I}_e$ denotes the Cauchy stress tensor of the exterior continuum, $\bar{h}_e$ denotes the heat flux vector, θ_e the temperature and $\dot{x}_e$ the material velocity of the exterior continuum. By $(\bar{I}_e)^\pm$ we understand limits when the crack-surface is approached from the interior of B_e . $r^\pm$ denotes the radiative heat-supply to the crack-surfaces from the environment.

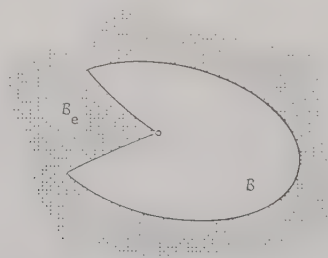


Figure 5.2

5.2 Moderate fracture-motions

A family of fracture-deformations $\chi(\cdot, t)$, $t \in T$ (satisfying K1 - K8, K9 (or K9', K9''), K10 and K13) is called a moderate fracture motion if

- a) there is a field defined on the crack-front $x'_F = x'_{F,t}(Z)$, $Z \in \tilde{F}_t$ such that

$$\lim_{\substack{X \rightarrow Z \\ X \in B \setminus \tilde{S}_t}} x'(X, t) = \lim_{\substack{X \rightarrow Z \\ X \in \tilde{S}_t \setminus \tilde{F}_t}} (x')^\pm(X, t) = x'_{F,t}(Z), \quad Z \in \tilde{F}_t$$

- b) $\lim_{\substack{X \rightarrow Z \\ X \in \tilde{S}_t \setminus \tilde{F}_t}} \tilde{\gamma}_t(X) \cdot \tilde{m}_t(X) (\tilde{F})^\pm(X, t) \tilde{m}_t(X) = \bar{0}$

Remark: This definition is motivated by the conjecture, put forward in section 2.3, that the field x' might be regular at the crack-front. Note that b) is trivially satisfied if the crack-surface $\tilde{S}_t$ is plane.

In preparation for our next proposition we introduce the energy-momentum tensors:

$$\begin{aligned} \underline{E}_p &= \underline{1}_V \epsilon \rho_p - \underline{F}^T \underline{I}_p, & \underline{\Psi}_p &= \underline{1}_V \psi \rho_p - \underline{F}^T \underline{I}_p \\ \underline{E}_{Sp}^\pm &= \underline{1}_T \epsilon_{Sp}^\pm - (\underline{F}^\pm)^T \underline{I}_{Sp}^\pm, & \underline{\Psi}_{Sp}^\pm &= \underline{1}_T \psi_{Sp}^\pm - (\underline{F}^\pm)^T \underline{I}_{Sp}^\pm \end{aligned} \quad (5.38)$$

Note that we have $\underline{\Psi}_p = \underline{E}_p - \underline{1}_V \theta \eta \rho_p$ and $\underline{\Psi}_{Sp}^\pm = \underline{E}_{Sp}^\pm - \underline{1}_T \theta_S^\pm \eta_{Sp}^\pm$.

Remark: We have the alternative representations

$$\underline{E}_p = \underline{1}_V \epsilon \rho_p - \underline{C} \underline{I}_{Sp} = \underline{J} \underline{E}$$

where $\underline{I}_{Sp} = \underline{J} \underline{F}^{-1} \underline{I} \underline{F}^{-T}$ is the second Piola-Kirchhoff stress tensor and

$$\underline{E} = \underline{1}_V \epsilon \rho - \underline{F}^T \underline{I} \underline{F}^{-T}$$

Proposition 5.5 In a moderate fracture-motion the local balance of momentum and moment of momentum at a crack-front material point are equivalent to respectively

$$\underline{J} \underline{S}_p[\tilde{\gamma}] = \lim_{R \rightarrow 0} \int_{\partial_t^-(R, s)} (\underline{I}_p - \underline{F} \tilde{\gamma} \otimes \tilde{\gamma} \tilde{C}^2 \rho_p) \tilde{e}_r \tilde{A} ds'(X) \quad (5.39)$$

and

$$\bar{0} = \lim_{R \rightarrow 0} \int_{Q_t(R,s)} (X - X^\pm(X(s,t), t)) \times (T_p - F \bar{\psi} \otimes \bar{\psi} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) \quad (5.40)$$

If, in addition

$$\int_{Q_t(R,s)} |T_p - F \bar{\psi} \otimes \bar{\psi} \bar{c}^2 \rho_p| \bar{e}_r \bar{A} ds'(X) \quad (5.41)$$

remains bounded as $R \rightarrow 0$, then the local balance of energy and entropy at a crack-front material point are equivalent to respectively

$$\begin{aligned} & \bar{\psi} \cdot [E_{Sp}[\bar{\psi} \bar{c} +] \bar{h}_{Sp}] \cdot \bar{\psi} \\ & \lim_{R \rightarrow 0} \int_{Q_t(R,s)} (\bar{h}_p \cdot \bar{e}_r + \bar{c} \bar{\psi} \cdot (E_p + \frac{1}{2} \bar{\psi} \cdot \bar{c} \bar{\psi} \bar{c}^2 \rho_p) \bar{e}_r) \bar{A} ds'(X) \quad (5.42) \end{aligned}$$

and

$$\begin{aligned} & (\bar{\psi} \cdot [\Psi_{Sp}[\bar{\psi} + \frac{1}{2} [\theta_S][n_{Sp}]] \bar{c} + \frac{1}{2} [\theta_S][\frac{1}{S} \bar{h}_{Sp}] \cdot \bar{\psi} + \{\theta_S\} \gamma_{Fp} = \\ & \lim_{R \rightarrow 0} \int_{Q_t(R,s)} ((1 - \frac{\{\theta_S\}}{\theta}) \bar{h}_p \cdot \bar{e}_r + \bar{c} \bar{\psi} \cdot (\Psi_p + \frac{1}{2} \bar{\psi} \cdot \bar{c} \bar{\psi} \bar{c}^2 + \\ & (1 - \frac{\{\theta_S\}}{\theta}) \theta n) \rho_p) \bar{e}_r) \bar{A} ds'(X) \quad (5.43) \end{aligned}$$

Proof: The equations (5.39) and (5.40) are direct consequences of the relation (cf. (2.76))

$$\dot{X} = X' - F \bar{\psi} \bar{c} \quad (5.44)$$

and the limits

$$\lim_{R \rightarrow 0} \int_{Q_t(R,s)} X' \rho_p \bar{c} \cdot \bar{e}_r \bar{A} ds'(X) = \bar{0} \quad (5.45)$$

$$\lim_{R \rightarrow 0} \int_{Q_t(R,s)} (X - X^\pm(X(s,t), t)) \times X' \rho_p \bar{c} \cdot \bar{e}_r \bar{A} ds'(X) = \bar{0}$$

From (5.44) it follows that

$$\dot{x} \cdot \dot{x} = x' \cdot x' + \bar{v} \cdot \bar{c} \bar{v} \bar{c}^2 - 2 x' \cdot \bar{c} \bar{v} \bar{c} \quad (5.46)$$

and then

$$\begin{aligned} \lim_{R \rightarrow 0} \int_{\bar{Q}_t(R,s)} (\bar{h}_p + \bar{I}_p^T \dot{x} + (\epsilon + \frac{1}{2} \dot{x} \cdot \dot{x}) \rho_p \bar{c}) \cdot \bar{e}_r \bar{A} ds'(X) = \\ \lim_{R \rightarrow 0} \int_{\bar{Q}_t(R,s)} (\bar{h}_p \cdot \bar{e}_r + \bar{c} \bar{v} \cdot (\bar{E}_p + \frac{1}{2} \bar{v} \cdot \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r + x' \cdot (\bar{I}_p - \\ \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r + \frac{1}{2} x' \cdot x' \rho_p \bar{c} \cdot \bar{e}_r) \bar{A} ds'(X) \end{aligned} \quad (5.47)$$

But

$$\lim_{R \rightarrow 0} \int_{\bar{Q}_t(R,s)} \frac{1}{2} x' \cdot x' \rho_p \bar{c} \cdot \bar{e}_r \bar{A} ds'(X) = 0 \quad (5.48)$$

Furthermore we may write

$$\begin{aligned} \int_{\bar{Q}_t(R,s)} x' \cdot (\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) = \\ x'_{F,t}(\hat{X}(s,t)) \cdot \int_{\bar{Q}_t(R,s)} (\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) + \\ \int_{\bar{Q}_t(R,s)} (x' - x'_{F,t}(\hat{X}(s,t))) \cdot (\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) \end{aligned} \quad (5.49)$$

and we have the simple estimate

$$\begin{aligned} \left| \int_{\bar{Q}_t(R,s)} (x' - x'_{F,t}(\hat{X}(s,t))) \cdot (\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) \right| \leq \\ \int_{\bar{Q}_t(R,s)} |(\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r| \bar{A} ds'(X) \sup_{X \in \bar{Q}_t(R,s)} |x'(X,t) - x'_{F,t}(\hat{X}(s,t))| \end{aligned} \quad (5.50)$$

By taking limits in (5.49) (as $R \rightarrow 0$) and using (5.39), (5.50) and the assumption that (5.41) remains bounded (as $R \rightarrow 0$) we may conclude that

$$\lim_{R \rightarrow 0} \int_{\bar{Q}_t(R,s)} x' \cdot (\bar{I}_p - \bar{c} \bar{v} \bar{c} \bar{v} \bar{c}^2 \rho_p) \bar{e}_r \bar{A} ds'(X) = x'_{F,t}(\hat{X}(s,t)) \cdot \int_{\bar{S}_p} \bar{v} \quad (5.51)$$

From (5.44) it follows that, at points on $\tilde{S}_t \setminus \tilde{F}_t$ (sufficiently close to the crack-front), we have

$$(x')^{\pm} = (\dot{x})^{\pm} + (\underline{E})^{\pm} \tilde{\nu} \underline{\zeta} \quad (5.52)$$

and by taking the scalar product of this equation with the vector $\underline{S}_p^{\pm} \tilde{\nu}_{||}$, where $\tilde{\nu}_{||} = \tilde{\nu} - \tilde{m} \cdot \tilde{\nu} \tilde{m}$, we get

$$(x')^{\pm} \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} = (\dot{x})^{\pm} \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} + ((\underline{E})^{\pm} \tilde{\nu}_{||}) \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} \underline{\zeta} + \tilde{m} \cdot \tilde{\nu} ((\underline{E})^{\pm} \tilde{m}) \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} \underline{\zeta}$$

and since $(\dot{x})^{\pm} = \dot{x}^{\pm}$ and $(\underline{E})^{\pm} \tilde{\nu}_{||} = \bar{\underline{E}}^{\pm} \tilde{\nu}_{||}$ (see Propositions 2.1 and 2.2) this may be written

$$(x')^{\pm} \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} = \dot{x}_{||}^{\pm} \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} + \tilde{\nu}_{||} \cdot (\bar{\underline{E}}^{\pm})^T \underline{S}_p^{\pm} \tilde{\nu}_{||} \underline{\zeta} + \tilde{m} \cdot \tilde{\nu} ((\bar{\underline{E}})^{\pm} \tilde{m}) \cdot \underline{S}_p^{\pm} \tilde{\nu}_{||} \underline{\zeta} \quad (5.53)$$

Finally, by taking limits in (5.53) (approaching the crack-front) we get the relation

$$x_F' \cdot \underline{S}_p^{\pm} \tilde{\nu} = \dot{x}_{||}^{\pm} \cdot \underline{S}_p^{\pm} \tilde{\nu} + \tilde{\nu} \cdot (\bar{\underline{E}}^{\pm})^T \underline{S}_p^{\pm} \tilde{\nu} \underline{\zeta}, \quad x \in \tilde{F}_t$$

and consequently

$$x_F' \cdot]\underline{S}_p^T[\tilde{\nu} =]\underline{S}_p^T \dot{x}[\cdot \tilde{\nu} + \tilde{\nu} \cdot]\bar{\underline{E}}^T \underline{S}_p[\tilde{\nu} \underline{\zeta}, \quad x \in \tilde{F}_t \quad (5.54)$$

From (5.47), (5.48), (5.51) and (5.54) it follows that

$$\begin{aligned} \lim_{R \rightarrow 0} \int_{\tilde{\Omega}_t(R,s)} (\bar{h}_p + \underline{I}_p^T \dot{x} + (\epsilon + \frac{1}{2} \dot{x} \cdot \dot{x}) \rho_p \tilde{c}) \cdot \tilde{e}_r \underline{A} ds'(X) = \\ \lim_{R \rightarrow 0} \int_{\tilde{\Omega}_t(R,s)} (\bar{h}_p \cdot \tilde{e}_r + \underline{\zeta} \tilde{\nu} \cdot (\underline{E}_p + \frac{1}{2} \underline{\zeta} \tilde{\nu} \underline{\zeta}^2 \rho_p) \tilde{e}_r) \underline{A} ds'(X) + \\]\underline{S}_p^T \dot{x}[\cdot \tilde{\nu} + \tilde{\nu} \cdot]\bar{\underline{E}}^T \underline{S}_p[\tilde{\nu} \underline{\zeta} \end{aligned} \quad (5.55)$$

The local balance of energy at a crack-front material point reads

$$]e_{Sp}[\underline{\zeta} +]\bar{h}_{Sp} + \underline{S}_p^T \dot{x}[\cdot \tilde{\nu} =$$

$$\lim_{R \rightarrow 0} \int_{\tilde{\Omega}_t(R,s)} (\bar{h}_p + \underline{I}_p^T \dot{x} + (\epsilon + \frac{1}{2} \dot{x} \cdot \dot{x}) \rho_p \tilde{c}) \cdot \tilde{e}_r \underline{A} ds'(X) \quad (5.56)$$

and from (5.55) it follows that this equation is equivalent to (5.42). The proof of (5.43) is equally simple. $\square$

Remark 1: Note that if (5.41) remains bounded as $R \rightarrow 0$ then (5.40) is satisfied automatically.

Remark 2: It follows from the proof of Proposition 5.5 that b), in the definition of a moderate fracture-motion, may be substituted for the somewhat weaker assumption:

$$\lim_{\substack{X \rightarrow Z \\ X \in \tilde{S}_t \setminus \tilde{F}_t}} \tilde{v}_t(X) \cdot \tilde{m}_t(X) \tilde{a}_t(X) \cdot (F)^\pm(X, t) \tilde{m}_t(X) = \bar{0} \quad (5.57)$$

for any C^0 -crack-surface-field $\tilde{a} = \tilde{a}_t(X) \in T_{X^\pm}^\pm(X, t)$.

Remark 3: We say that steady state conditions for X are prevailing at the crack-front if (cf. section 2.3)

$$\begin{aligned} x'(X, t) &= \bar{0} \\ \dot{\tilde{c}}_t(X) &= \bar{0} \quad X \in \tilde{N}_t(r_0) \setminus \tilde{S}_t \end{aligned}$$

Then of course b) is satisfied with $x'_F = \bar{0}$. Note that, at steady state, the local balance of momentum at a regular bulk material point may be written (see Proposition 2.13)

$$(\tilde{c}^{2,2} F(\tilde{v}, \tilde{v}) - \bar{b}) \rho_p - \text{Div } \tilde{I}_p = \bar{0} \quad (5.58)$$

and if $\bar{b}$ is assumed to be bounded and ${}^2F(\tilde{v}, \tilde{v})$ is singular at the crack-front then asymptotically (as $\tilde{F}$ is approached from $B \setminus \tilde{F}$) we have

$$\tilde{c}^{2,2} F(\tilde{v}, \tilde{v}) \rho_p - \text{Div } \tilde{I}_p \approx \bar{0} \quad (5.59)$$

If the limit

$$\bar{\alpha} \equiv \lim_{R \rightarrow 0} \int_{\tilde{Q}_t(R, s)} \tilde{I}_p \tilde{e}_r \tilde{A} ds'(X) \quad (5.60)$$

exists, then the balance of momentum, (5.39), provides a formula for the calculation of the crack-front speed when the stress-conditions at the crack-front are known:

Corollary 5.3

$$\underline{\zeta} = \pm \sqrt{\frac{|\bar{\alpha} -]S_p[\underline{\tilde{\gamma}}|}{|\bar{\beta}|}} \quad (5.61)$$

where

$$\bar{\beta} = \lim_{R \rightarrow 0} \int_{\underline{\mathcal{Q}}_t(R,t)} F \underline{\tilde{\gamma}} \cdot \underline{\tilde{e}}_r \rho_p \underline{\tilde{\gamma}} ds'(X) \neq \bar{0} \quad (5.62)$$

Proof: This is an immediate consequence of (5.39). $\square$

Remark: If $\bar{\beta} = \bar{0}$ then the local balance of momentum at a crack-front material point is equivalent to

$$]S_p[\underline{\tilde{\gamma}} = \bar{\alpha} \quad (5.63)$$

We say that a moderate fracture-motion has quasi-static conditions at the crack-front if

$$\lim_{R \rightarrow 0} \int_{\underline{\mathcal{Q}}_t(R,s)} \underline{\tilde{\gamma}} \cdot \underline{\tilde{C}} \underline{\tilde{\gamma}} ds'(X) = 0 \quad (5.64)$$

Proposition 5.6 In a moderate fracture-motion with quasi-static conditions at the crack-front the local balance of momentum and moment of momentum at a crack-front material point are equivalent to respectively

$$]S_p[\underline{\tilde{\gamma}} = \lim_{R \rightarrow 0} \int_{\underline{\mathcal{Q}}_t(R,s)} T_p \underline{\tilde{e}}_r \underline{\tilde{\gamma}} ds'(X) \quad (5.65)$$

and

$$\bar{0} = \lim_{R \rightarrow 0} \int_{\underline{\mathcal{Q}}_t(R,s)} (X - X^\pm(\hat{X}(s,t),t)) \times T_p \underline{\tilde{e}}_r \underline{\tilde{\gamma}} ds'(X) \quad (5.66)$$

If, in addition

$$\int_{\underline{\mathcal{Q}}_t(R,s)} |T_p \underline{\tilde{e}}_r| \underline{\tilde{\gamma}} ds'(X) \quad (5.67)$$

remains bounded as $R \rightarrow 0$ and if we have homothermal conditions at the

crack-front (cf. section 5.1), then the local balance of energy and entropy at a crack-front material point are equivalent to respectively

$$\begin{aligned} \bar{\gamma} \cdot]E_{Sp}[\bar{\gamma} \bar{\gamma} +]\bar{h}_{Sp}[\cdot \bar{\gamma} = \\ \lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R,s)} (\bar{h}_p \cdot \bar{e}_r + \bar{\gamma} \bar{\gamma} \cdot E_p \bar{e}_r) \bar{\Delta} ds'(X) \end{aligned} \quad (5.68)$$

and

$$\bar{\gamma} (\bar{\gamma} \cdot]\Psi_{Sp}[\bar{\gamma} - \lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R,s)} \bar{\gamma} \cdot \Psi_p \bar{e}_r \bar{\Delta} ds'(X)) + \theta_F \gamma_{Fp} = 0 \quad (5.69)$$

Proof: By using the Cauchy-Schwartz inequality we obtain the simple estimate

$$\begin{aligned} \left| \int_{\bar{\mathcal{Q}}_t(R,s)} E \bar{\gamma} \bar{\gamma} \cdot \bar{e}_r \bar{\gamma}^2 \rho_p \bar{\Delta} ds'(X) \right| \leq C (\bar{\gamma}_t(s))^2 (1 + R \bar{\kappa}_t(s)) \int_{\bar{\mathcal{Q}}_t(R,s)} |E \bar{\gamma}| ds'(X) \\ \leq C (\bar{\gamma}_t(s))^2 (1 + R \bar{\kappa}_t(s)) \sqrt{2\pi R} \sqrt{\int_{\bar{\mathcal{Q}}_t(R,s)} \bar{\gamma} \cdot C \bar{\gamma} ds'(X)} \end{aligned} \quad (5.70)$$

where C is a constant satisfying $C \geq \sup_{X \in B} \rho_p(X)$. From (5.64) and (5.70) we may conclude that

$$\lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R,s)} E \bar{\gamma} \bar{\gamma} \cdot \bar{e}_r \bar{\gamma}^2 \rho_p \bar{\Delta} ds'(X) = 0 \quad (5.71)$$

The equations (5.65) and (5.66) now follow from (5.71) and Proposition 5.5. It is evident, from (5.67) and (5.71), that (5.41) remains bounded as $R \rightarrow 0$. The equations (5.68) and (5.69) are then direct consequences of Proposition 5.5 and (5.64). $\square$

Remark: It is apparent from (5.70) that for the proof of (5.65) and (5.66) it is sufficient to assume that (cf. (5.64))

$$\lim_{R \rightarrow 0} R \int_{\bar{\mathcal{Q}}_t(R,s)} \bar{\gamma} \cdot C \bar{\gamma} ds'(X) = 0 \quad (5.72)$$

A few simple consequences of Proposition 5.6 are collected in the following Corollary.

Corollary 5.4 Let $x(\cdot, t)$, $t \in T$ be a moderate fracture-motion with quasi-static and homothermal conditions at the crack-front

b) If the crack is resting ($\underline{c} = 0$) then the local balance of energy and entropy at a crack-front material point are equivalent to respectively

$$\bar{h}_{Sp}[\cdot, \bar{\psi}] = \lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R, s)} \bar{h}_p \cdot \bar{e}_r \bar{A} ds'(X) \quad (5.73)$$

and

$$\gamma_{Fp} = 0 \quad (5.74)$$

b) If the crack is advancing (or retreating, $\underline{c} \neq 0$) and if we have global homothermal conditions (cf. section 5.1) then the local balance of energy at a crack-front material point is equivalent to

$$\bar{\psi} \cdot \bar{E}_{Sp}[\bar{\psi}] = \lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R, s)} \bar{\psi} \cdot \bar{E}_p \bar{e}_r \bar{A} ds'(X) \quad (5.75)$$

Proof: These equations are direct consequences of Proposition 5.6. $\square$

Remark: In b) the production of entropy on the crack-front may be written

$$\begin{aligned} \gamma_{Fp} &= \frac{\underline{c}}{\theta} \left(\lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R, s)} \bar{\psi} \cdot \bar{\Psi}_p \bar{e}_r \bar{A} ds'(X) - \bar{\psi} \cdot \bar{\Psi}_{Sp}[\bar{\psi}] \right) = \\ &\underline{c} (\bar{\eta}_{Sp} - \lim_{R \rightarrow 0} \int_{\bar{\mathcal{Q}}_t(R, s)} \eta_p \bar{\psi} \cdot \bar{e}_r \bar{A} ds'(X)) \end{aligned} \quad (5.76)$$

5.3 Path-independent integrals of Rice and Sih

Consider a body B of homogeneous, thermoelastic bulk material with thermodynamic potential $\hat{\psi} = \hat{\psi}(F, \theta)$. That is

$$\begin{aligned} \psi(X, t) &= \hat{\psi}(F(X, t), \theta(X, t)), \quad T_p(X, t) = \rho_p(X) \frac{\partial \hat{\psi}}{\partial F}(F(X, t), \theta(X, t)) \\ \eta(X, t) &= - \frac{\partial \hat{\psi}}{\partial \theta}(F(X, t), \theta(X, t)) \end{aligned} \quad (5.77)$$

See for instance Suhubi [16].

Proposition 5.7 Let P be a non-cracking subbody of B . Then

$$\int_{\partial P} \Psi_P \bar{n} da(X) = \int_{\sum_t \cap P} ([\psi \rho_p] - \{I_p\} \cdot [E]) \bar{M} + \{E\}^T [\dot{x}] \mu_p da(X) + \int_B (-\bar{g} \eta \rho_p + \psi \text{Grad } \rho_p - E^T (\ddot{x} - \bar{b}) \rho_p) dv(X) \quad (5.78)$$

where $\bar{g} \equiv \text{Grad } \theta$ and $\{I_p\}$ is defined in (5.23).

Proof: By using the divergence theorem we get

$$\int_{\partial P} \Psi_P \bar{n} da(X) = \int_{\sum_t \cap P} [\Psi_P] \bar{M} da(X) + \int_B \text{Div } \Psi_P dv(X) \quad (5.79)$$

Clearly we have

$$[\Psi_P] \bar{M} = [\psi \rho_p] \bar{M} - [E^T I_p] \bar{M} = [\psi \rho_p] \bar{M} - [E]^T \{I_p\} \bar{M} - \{E\}^T [I_p] \bar{M} \quad (5.80)$$

But, since $[E] = \bar{f} \otimes \bar{M}$, $\bar{f} \equiv [(\nabla X) \bar{M}]$ (cf. Truesdell & Toupin [5], p. 494, eq. (175.9)), we get

$$[E]^T \{I_p\} \bar{M} = ((\{I\}^T \bar{f}) \cdot \bar{M}) \bar{M} = \{I_p\} \cdot [E] \bar{M} \quad (5.81)$$

Further, from the local balance of momentum (at a singular bulk material point), we get

$$\{E\}^T [I_p] \bar{M} = - \{E\}^T [\dot{x}] \mu_p \quad (5.82)$$

By a simple calculation, using (5.77) and the local balance of momentum (at a regular bulk material point), we obtain the relation

$$\text{Div } \Psi_P = -\bar{g} \eta \rho_p + \psi \text{Grad } \rho_p - E^T (\ddot{x} - \bar{b}) \rho_p \quad (5.83)$$

Formula (5.78) now follows from (5.79), (5.81) - (5.83). $\square$

The reference mass density is said to be homogeneous if

$$\text{Grad } \rho_p = \bar{0}, \quad X \in B \quad (5.84)$$

We say that we have global quasi-static conditions in B if

$$\begin{aligned}\ddot{\mathbf{x}} - \ddot{\mathbf{b}} &= \ddot{\mathbf{0}}, & \mathbf{x} \in B \setminus \Sigma_t \\ [\dot{\mathbf{x}}] &= \ddot{\mathbf{0}}, & \mathbf{x} \in \Sigma_t\end{aligned}\quad (5.85)$$

and if we have quasi-static conditions at the crack-front. We only consider moderate fracture-motions in this section.

Corollary 5.5 Let P be a non-cracking subbody of B with homogeneous reference mass density. If we have global quasi-static and global homothermal conditions and if

$$[\psi \rho_p] - \{\mathbf{I}_p\} \cdot [\underline{\mathbf{E}}] = 0, \quad \mathbf{x} \in \Sigma_t \quad (5.86)$$

then

$$\int_{\partial P} \underline{\psi} \cdot \underline{\tilde{n}} \, da(\mathbf{x}) = \ddot{\mathbf{0}} \quad (5.87)$$

Proof: This is a direct consequence of formula (5.78). $\square$

Remark 1: Formula (5.87) is called Eshelby's conservation law. Eshelby, in a paper devoted to the continuum theory of lattice defects, introduced the integral

$$\int_{\partial P} \underline{\psi} \cdot \underline{\tilde{n}} \, da(\mathbf{x})$$

as "a generalized force on an elastic singularity" (see [17] and [18]).

Remark 2: From (5.85)₂ it follows that $[\mathbf{I}_p] \underline{\tilde{m}} = \ddot{\mathbf{0}}$. Consequently $\{\mathbf{I}_p\} \underline{\tilde{m}} = (\mathbf{I}_p)^\pm \underline{\tilde{m}}$ and (5.86) may be written

$$[\psi \rho_p] - (\mathbf{I}_p)^\pm \cdot [\underline{\mathbf{E}}] = 0$$

or in terms of the thermodynamic potential

$$\hat{\psi}((\underline{\mathbf{E}})^+) - \hat{\psi}((\underline{\mathbf{E}})^-) = \frac{\partial \hat{\psi}}{\partial \underline{\mathbf{E}}}((\underline{\mathbf{E}})^\pm) \cdot ((\underline{\mathbf{E}})^+ - (\underline{\mathbf{E}})^-) \quad (5.88)$$

This is called the Maxwell relation (see Gurtin [19]).

We now assume that B contains a plane crack-surface $\tilde{S}_t$ and a straight-lined crack-front $\tilde{F}_t$ in the referential description. Let $\tilde{T}_{\delta_t}$ be a tubular surface (surrounding the crack-front) with cross-sectional curves $\tilde{\Gamma}_t = \tilde{\Gamma}_t(s)$ (cf. section 2.3). It is assumed that the shape of $\tilde{\Gamma}_t$ is independent of s (that is, the defining function in (2.63) is independent of s). We also assume that B has homogeneous reference mass density.

Proposition 5.8 If, in addition to the assumptions stated above, we have global quasi-static and global homothermal conditions in B and if

- a) $[\psi \rho_p] - \{\mathbf{T}_p\} \cdot [\mathbf{F}] = 0, \quad X \in \tilde{\Sigma}_t$
- b) $\bar{t}_{ep}^\pm + \text{Div}_{\delta} \mathbf{S}_p^\pm = \bar{0}, \quad X \in \tilde{S}_t \setminus \tilde{F}_t$
- c) $\bar{\mathbf{v}} \cdot \mathbf{T}_p \bar{\mathbf{e}} = 0, \quad X \in \tilde{B} \setminus \tilde{S}_t$

where $\bar{\mathbf{e}}$ is the unit tangent vector on $\tilde{F}$ and $\bar{\mathbf{v}} \equiv \mathbf{F}^* \tilde{\mathbf{v}}$ (cf. (2.79)), then

$$\lim_{R \rightarrow 0} \int_{\tilde{Q}_t(R,s)} \tilde{\mathbf{v}} \cdot \Psi_p \bar{\mathbf{e}}_r ds_0(X) = \int_{\tilde{\Gamma}_t(s)} \tilde{\mathbf{v}} \cdot \Psi_p \bar{\mathbf{N}} ds_\Gamma(X) \quad (5.89)$$

where $\bar{\mathbf{N}} = \bar{\mathbf{N}}_t(X)$ is the outward pointing (relative to $\text{int}(\tilde{T}_{\delta_t})$) unit normal vector on $\tilde{T}_{\delta_t}$ (ds_0 and ds_Γ denote arc-length measures on $\tilde{Q}_t$ and $\tilde{\Gamma}_t$ respectively).

Proof: Let $\tilde{U}_t(R)$ denote a circular tube surrounding the crack-front in the referential description. Consider the region $\mathcal{D}$ bounded by $\tilde{T}_{\delta_t}$, $\tilde{U}_t(R)$ (R sufficiently small) and two tubular cross-sections separated by a segment $\tilde{F}_t^-$ of the crack-front.

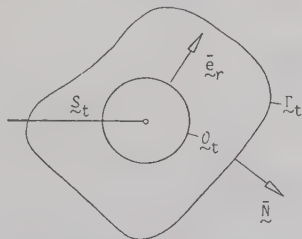


Figure 5.3

Consider the field $\Psi_p^T \tilde{\mathbf{v}}$. From b) and the local balance of momentum (at a

crack-surface material point) we get $(\mathbb{T}_p)^+ \bar{\mathbb{m}} = \bar{0}$. Thus

$$[\Psi_p^T \bar{\mathbb{v}}] \cdot \bar{\mathbb{m}} = 0, \quad X \in \underline{S}_t \setminus \underline{F}_t \quad (5.90)$$

From a) we obtain

$$[\Psi_p^T \bar{\mathbb{v}}] \cdot \bar{\mathbb{e}} = 0, \quad X \in B \setminus \underline{S}_t \quad (5.91)$$

An application of the divergence theorem to the field $\Psi_p^T \bar{\mathbb{v}}$ and the region D gives

$$\int_D \text{Div}(\Psi_p^T \bar{\mathbb{v}}) dv(X) = - \int_{\underline{U}_t(R)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{e}}_r da(X) + \int_{\underline{S}_t} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{N}} da(X) \quad (5.92)$$

where we have used (5.90) and (5.91). Here $\underline{U}_t(R)$ and $\underline{S}_t$ denote those parts of $\underline{U}_t(R)$ and $\underline{S}_t$ respectively that correspond to the crack-front segment $\underline{F}_t$. But (cf. (5.83))

$$\text{Div}(\Psi_p^T \bar{\mathbb{v}}) = \bar{\mathbb{v}} \cdot \text{Div} \Psi_p = \bar{0} \quad (5.93)$$

and (5.92) may be written (the area factors are equal to one)

$$\int_{\underline{F}_t} \left(\int_{\underline{Q}_t(R,s)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{e}}_r ds_0(X) - \int_{\underline{\Gamma}_t(s)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{N}} ds_\Gamma(X) \right) ds(X) = 0 \quad (5.94)$$

From the arbitrariness of the crack-front segment $\underline{F}_t$ and the continuity of the integrand we conclude that

$$\int_{\underline{Q}_t(R,s)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{e}}_r ds_0(X) = \int_{\underline{\Gamma}_t(s)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{N}} ds_\Gamma(X)$$

and this proves the Proposition. $\square$

Remark 1: The integral

$$J = J_t(s) \equiv \int_{\underline{\Gamma}_t(s)} \bar{\mathbb{v}} \cdot \Psi_p \bar{\mathbb{N}} ds_\Gamma(X) \quad (5.95)$$

is the celebrated path-independent J-integral introduced in Rice [20]

Remark 2: Note that if c) is changed for the somewhat stronger assumption

$$\mathbb{T}_p \bar{\mathbb{e}} = \bar{0}, \quad X \in B \setminus \underline{S}_t \quad (5.96)$$

then, by using the same technique as above, we obtain

$$\lim_{R \rightarrow 0} \int_{Q_t(R,s)} \mathbb{T}_p \tilde{\mathbf{e}}_r ds_O(X) = \int_{\Gamma_t(s)} \mathbb{T}_p \tilde{\mathbf{N}} ds_\Gamma(X) \quad (5.97)$$

Let us now assume that, instead of quasi-static conditions, we have steady state conditions in B . If $\bar{\mathbf{b}} = \bar{\mathbf{0}}$ then the local balance of momentum at a regular bulk material point reads

$$\tilde{\mathbf{c}}^2 {}^2\mathbb{F}(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}) \rho_p - \text{Div } \mathbb{T}_p = \bar{\mathbf{0}} \quad (5.98)$$

A simple calculation gives ($\tilde{\mathbf{v}}$ is independent of X)

$$\text{Grad}(\tilde{\mathbf{v}} \cdot \mathbb{C} \tilde{\mathbf{v}}) = 2 ({}^2\mathbb{F}_{\tilde{\mathbf{v}}}^T)^T \mathbb{F} \tilde{\mathbf{v}} \quad (5.99)$$

(The tensor ${}^2\mathbb{F}_{\tilde{\mathbf{v}}} = {}^2\mathbb{F}_{\tilde{\mathbf{v}}}(X, t) \in \text{End}(V)$ is defined by: ${}^2\mathbb{F}_{\tilde{\mathbf{v}}} \bar{\mathbf{a}} \equiv {}^2\mathbb{F}(\bar{\mathbf{a}}, \tilde{\mathbf{v}})$, $\bar{\mathbf{a}} \in V$).

If we have homothermal conditions and if ρ_p and $\tilde{\mathbf{c}}$ are independent of X then, from (5.77), (5.98) and (5.99), we get Sih's relation (cf. [21])

$$\text{Div}(\Psi_p^T \tilde{\mathbf{v}} + \frac{1}{2} \tilde{\mathbf{v}} \cdot \mathbb{C} \tilde{\mathbf{v}} \tilde{\mathbf{c}}^2 \rho_p \tilde{\mathbf{v}}) = 0 \quad (5.100)$$

We may now prove, using the technique presented in the proof of Proposition 5.8, the path-independent representation

$$\begin{aligned} \lim_{R \rightarrow 0} \int_{Q_t(R,s)} \tilde{\mathbf{v}} \cdot (\Psi_p + \frac{1}{2} \tilde{\mathbf{v}} \cdot \mathbb{C} \tilde{\mathbf{v}} \tilde{\mathbf{c}}^2 \rho_p) \tilde{\mathbf{e}}_r ds_O(X) = \\ \int_{\Gamma_t(s)} \tilde{\mathbf{v}} \cdot (\Psi_p + \frac{1}{2} \tilde{\mathbf{v}} \cdot \mathbb{C} \tilde{\mathbf{v}} \tilde{\mathbf{c}}^2 \rho_p) \tilde{\mathbf{e}}_r ds_\Gamma(X) \end{aligned} \quad (5.101)$$

APPENDIX

This Appendix presents some auxiliary topics in algebra and analysis that are used in the main text. For proofs and additional details the reader is referred to the relevant mathematical literature. See also chapter 1 for some basic notation.

A.1 Reciprocal basis

Let V be a 3-dimensional, oriented Euclidean vector-space with scalar-product " $\cdot$ " and vector-product " $\times$ ". Let $(\bar{a}_1 \bar{a}_2 \bar{a}_3)$ be a basis for V . The reciprocal basis $(\bar{a}^1 \bar{a}^2 \bar{a}^3)$ relative to $(\bar{a}_1 \bar{a}_2 \bar{a}_3)$ is defined by

$$\bar{a}^1 = \frac{1}{A} \bar{a}_2 \times \bar{a}_3, \quad \bar{a}^2 = \frac{1}{A} \bar{a}_3 \times \bar{a}_1, \quad \bar{a}^3 = \frac{1}{A} \bar{a}_1 \times \bar{a}_2 \quad (\text{A.1})$$

where $A \equiv (\bar{a}_1 \times \bar{a}_2) \cdot \bar{a}_3$. See Bowen & Wang [22].

A.2 Polar decomposition theorem

Each $\underline{F} \in \text{IL}(U, W)$ has a unique factorization

$$\underline{F} = \underline{R} \underline{U} \quad (\text{A.2})$$

with $\underline{U} \in \text{Sym}^+(U)$ and $\underline{R} \in O(U, W)$. Furthermore $\underline{U}^2 = \underline{F}^T \underline{F}$.

Proof: See Halmos [23] or Bowen & Wang [22]. $\square$

A.3 Differentiating a determinant depending on a parameter

Let $T \ni t \mapsto \underline{A}(t) \in \text{Aut}(V)$ be a smooth function. Then

$$\frac{d}{dt} \det \underline{A} = \det \underline{A} \operatorname{tr} \left(\underline{A}^{-1} \frac{d\underline{A}}{dt} \right) \quad (\text{A.3})$$

Proof: Let V be a 3-dimensional, oriented Euclidean vector-space. The determinant is characterized by the relation

$$(\underline{A} \bar{a} \times \underline{A} \bar{b}) \cdot \underline{A} \bar{c} = \det \underline{A} (\bar{a} \times \bar{b}) \cdot \bar{c}, \quad \bar{a}, \bar{b}, \bar{c} \in V \quad (\text{A.4})$$

The result is now obtained by differentiating (A.4) (for fixed $\bar{a}, \bar{b}, \bar{c}$) and observing that the trace is characterized by the relation

$$((\underline{A} \bar{a}) \times \bar{b}) \cdot \bar{c} + (\bar{a} \times \underline{A} \bar{b}) \cdot \bar{c} + (\bar{a} \times \bar{b}) \cdot \underline{A} \bar{c} = \text{tr } \underline{A} (\bar{a} \times \bar{b}) \cdot \bar{c} \quad (\text{A.5})$$

See Bowen & Wang [22]. $\square$

A.4 Global inverse function theorem

Let E, E' denote finite-dimensional Euclidean point spaces with corresponding translation spaces V, V' . Let $\mathcal{D}$ be an open set in E and let $f: \mathcal{D} \rightarrow f(\mathcal{D}) \subset E'$ be a smooth mapping satisfying:

- (i) $\nabla f(x) \in \text{IL}(V, V')$ for every $x \in \mathcal{D}$
- (ii) there is a constant C such that $|\nabla f(x)|^{-1} \leq C$ for every $x \in \mathcal{D}$.

Then f is a diffeomorphism.

Proof: See Choquet-Bruhat et. al. [24]. $\square$

A.5 Curves in E

Let E be a 3-dimensional, oriented Euclidean point space. Let C be a smooth curve in E with a global parametrization n

$$E \supset C \ni x = \hat{x}(s), \quad s \in [s_1, s_2]$$

where s is the arc-length coordinate on C . The unit tangent $\bar{e} = \bar{e}(s)$, the principal normal $\bar{p} = \bar{p}(s)$ and the bi-normal $\bar{b} = \bar{b}(s)$ are defined by

$$\bar{e}(s) \equiv \frac{d\hat{x}}{ds}(s), \quad \bar{p}(s) \equiv \frac{1}{\kappa(s)} \frac{d\bar{e}}{ds}(s), \quad \bar{b}(s) \equiv \bar{e}(s) \times \bar{p}(s) \quad (\text{A.6})$$

where $\kappa = \kappa(s) \equiv |\frac{d\bar{e}}{ds}(s)|$ is the curvature of C . Derivatives are obtained according to Frenet's formula:

$$\frac{d\bar{a}}{ds}(s) = \bar{\delta}(s) \times \bar{a}(s), \quad \bar{a} = \bar{e}, \bar{p}, \bar{b} \quad (\text{A.7})$$

$$\bar{\delta}(s) \equiv \bar{e}(s) \tau(s) + \bar{b}(s) \kappa(s) \quad (\text{"the Darboux vector"})$$

where $\tau = \tau(s)$ is the torsion of C .

A moving curve in E is a family of smooth curves $C_t, t \in T$ (T an in-

terval in $\mathbb{R}$)

$$C_t \ni x = \hat{x}(q, t), \quad q \in I$$

where we assume that $\hat{x}(q, \cdot) : T \rightarrow E$ is a smooth mapping. We may then define the normal velocity $\bar{c}_t$ of C_t

$$\bar{c}_t \equiv \frac{\partial \hat{x}}{\partial t} - \bar{e}_t (\bar{e}_t \cdot \frac{\partial \hat{x}}{\partial t}) \quad (\text{A.8})$$

where $\bar{e}_t = \bar{e}_t(q)$ is the unit tangent on C_t . Note that this definition is independent of the particular parametrization $\hat{x}(\cdot, t) : I \rightarrow C_t$.

A.6 Surfaces in E^1

Let E be a 3-dimensional, oriented Euclidean point space with translation space V .

A smooth surface-with-boundary in E is a set $S \subset E$ with the property that every $x \in S$ has a neighbourhood (in the relative topology on S) homeomorphic to an open subset U of $\mathbb{R}_+^2 \equiv \{(q_1, q_2) \mid q_2 \geq 0\}$. This homeomorphism $\hat{x} : U \rightarrow S$ is called a local parametrization $(U, \hat{x})$ of S . It is assumed that $\hat{x}$ is smooth mapping and that $\nabla \hat{x}(q_1, q_2) : \mathbb{R}^2 \rightarrow V$ is injective. The boundary of S , being a smooth curve in E , is denoted ∂S . The tangent space of S at x , $T_x(S)$, is the 2-dimensional subspace of V defined by

$$T_x(S) \equiv \nabla \hat{x}(q_1, q_2)(\mathbb{R}^2), \quad x = \hat{x}(q_1, q_2) \quad (\text{A.9})$$

The inclusion-mapping $\underline{I}(x) \in L(T_x, V)$ is defined by

$$\underline{I}(x) \bar{u} = \bar{u}, \quad \bar{u} \in T_x \quad (\text{A.10})$$

Let $\bar{v} \in V$, then $\bar{v}$ admits a unique decomposition

$$\bar{v} = \bar{v}_{||} + \bar{v}_{\perp}$$

with $\bar{v}_{||} \in T_x$ and $\bar{v}_{\perp} \in T_x^{\perp}$ (the orthogonal complement of T_x in V). The perpendicular projection $\underline{P}(x) \in L(V, T_x)$ is defined by

$$\underline{P}(x) \bar{v} = \bar{v}_{||} \quad (\text{A.11})$$

¹ We have adopted some of the notation in [13].

The inner product on V induces an inner product on the subspace T_x in the natural way. Then for every $\bar{u} \in T_x$, $\bar{v} \in V$ we have $\bar{u} \cdot \underline{P}(x) \bar{v} = (\underline{I}(x) \bar{u}) \cdot \bar{v}$ so that

$$\underline{I}(x)^T = \underline{P}(x) \quad (\text{A.12})$$

Furthermore

$$\underline{P}(x) \underline{I}(x) = \underline{I}_{T_x} \quad (\text{A.13})$$

where $\underline{I}_{T_x}$ is the identity mapping on T_x , the metric tensor on S .

Given two smooth surfaces S_1, S_2 in E . Let O_1 be an open subset in S_1 .

A mapping $f: O_1 \rightarrow S_2$ is said to be differentiable at $x \in O_1$ if, in parametrizations $(U_1, \hat{x}_1)$ at x and $(U_2, \hat{x}_2)$ at $f(x)$, the mapping $\hat{x}_2^{-1} \circ f \circ \hat{x}_1: U_1 \rightarrow U_2$ is differentiable at the point $\hat{x}_1^{-1}(x)$.

The surface gradient $\nabla_{\Delta} f(x) \in L(T_x(S_1), T_{f(x)}(S_2))$ is defined by

$$\nabla_{\Delta} f \equiv (\nabla \hat{x}_2) \nabla(\hat{x}_2^{-1} \circ f \circ \hat{x}_1)(\nabla \hat{x}_1)^{-1} \quad (\text{A.14})$$

Given a smooth surface S and an open set $O \subset S$. A mapping $f: O \rightarrow \mathbb{R}, V, \text{End}(V)$ is said to be differentiable at $x \in O$ if, in a parametrization $(U, \hat{x})$ at x , the mapping $f \circ \hat{x}: U \rightarrow \mathbb{R}, V, \text{End}(V)$ is differentiable.

The surface gradient $\nabla_{\Delta} f(x) \in L(T_x, \mathbb{R})$ (or $L(T_x, V), L(T_x, \text{End}(V))$) is defined by

$$\nabla_{\Delta} f \equiv \nabla(f \circ \hat{x}_1)(\nabla \hat{x}_1)^{-1} \quad (\text{A.15})$$

Let f be a field, defined on S , that assigns to each $x \in O$ a linear transformation $f(x) \in L(T_x, V)$. Consider the field $f_p: O \rightarrow \text{End}(V)$ defined by $f_p(x) \equiv f(x) \underline{P}(x)$ where $\underline{P}$ is the projection operator on S . We say that f is differentiable if f_p is differentiable in the sense described above. In the same manner we say that f , with $f(x) \in \text{End}(T_x)$, is differentiable if $f_{ip}: O \rightarrow \text{End}(V)$, defined by $f_{ip}(x) \equiv \underline{I}(x) f(x) \underline{P}(x)$, is differentiable in the sense described above.

A surface S is said to be orientable if there exists a smooth vector-field $\bar{n} \equiv \bar{n}(x)$ on S such that $\bar{n}(x) \in T_x^{\perp}$, $|\bar{n}(x)| = 1$. The field $\bar{n}$ is called an orientation (a unit normal vector field) on S . We have the obvious relation

$$\underline{I}(x) \underline{P}(x) = \underline{I}_{T_x} - \bar{n}(x) \otimes \bar{n}(x), \quad x \in S \quad (\text{A.16})$$

The tangential surface gradient $D\bar{u}$ of a smooth vector-field $\bar{u}: S \rightarrow V$ is defined by

$$D\bar{u} \equiv \underline{P} \nabla_{\delta} \bar{u} \quad (\text{A.17})$$

so that $D\bar{u}(x) \in \text{End}(T_x)$.

Let S be a surface with orientation $\bar{n}$. The Weingarten mapping $\underline{L}$ for the surface is defined by

$$\underline{L} \equiv -D\bar{n} \quad (\text{A.18})$$

so that $\underline{L}(x) \in \text{Sym}(T_x)$. Note that $\underline{I}\underline{L} = -\nabla_{\delta} \bar{n}$.

$$M \equiv \frac{1}{2} \text{tr} \underline{L} \quad \text{and} \quad G \equiv \det \underline{L} \quad (\text{A.19})$$

are called the mean and Gaussian curvature respectively.

The surface divergence $\text{div}_{\delta} \bar{u}$ of a smooth vector-field $\bar{u}: S \rightarrow V$ is defined by

$$\text{div}_{\delta} \bar{u} \equiv \text{tr} D\bar{u} \quad (\text{A.20})$$

Let $\underline{A}$ be a smooth field, defined on S , that assigns to each x a linear transformation $\underline{A}(x) \in L(T_x, V)$. The surface divergence $\text{div}_{\delta} \underline{A}$ is the vector-field on S defined by

$$(\text{div}_{\delta} \underline{A}) \cdot \bar{v} = \text{div}_{\delta} (\underline{A}^T \bar{v}), \quad \text{for all } \bar{v} \in V \quad (\text{A.21})$$

Let $\underline{A}$ be a smooth field, defined on S , that assigns to each x a linear transformation $\underline{A}(x) \in \text{End}(T_x)$. The surface-divergence $\text{div}_{\delta} \underline{A}$ is the vector-field on S defined by

$$\text{div}_{\delta} \underline{A} \equiv \text{div}_{\delta} (\underline{I} \underline{A}) \quad (\text{A.22})$$

or equivalently

$$(\text{div}_{\delta} \underline{A}) \cdot \bar{v} = \text{div}_{\delta} (\underline{A}^T \underline{P} \bar{v}), \quad \text{for all } \bar{v} \in V$$

Let f, g be scalar-fields, $\bar{u}, \bar{v}$ vector-fields and $\underline{A}$, with $\underline{A}(x) \in L(\tau_x, \nu)$, a tensor-field defined on S . Among the rules for taking surface-gradients and surface-divergences of products the following are used in the main text:

$$\begin{aligned}
 \nabla_{\Delta}(fg) &= g \nabla_{\Delta} f + f \nabla_{\Delta} g \\
 \nabla_{\Delta}(\bar{u} \cdot \bar{v}) &= (\nabla_{\Delta} \bar{u})^T \bar{v} + (\nabla_{\Delta} \bar{v})^T \bar{u} \\
 \nabla_{\Delta}(f \bar{u}) &= f \nabla_{\Delta} \bar{u} + \bar{u} \otimes \nabla_{\Delta} f \\
 \operatorname{div}_{\Delta}(f \bar{u}) &= f \operatorname{div}_{\Delta} \bar{u} + \bar{u} \cdot \nabla_{\Delta} f \\
 \operatorname{div}_{\Delta}(f \underline{A}) &= f \operatorname{div}_{\Delta} \underline{A} + \underline{A} \nabla_{\Delta} f \\
 \operatorname{div}_{\Delta}(\underline{A}^T \bar{u}) &= (\operatorname{div}_{\Delta} \underline{A}) \cdot \bar{u} + \underline{A} \cdot \nabla_{\Delta} \bar{u} \\
 \operatorname{div}_{\Delta}(\bar{u} \times \underline{A}) &= \bar{u} \times \operatorname{div}_{\Delta} \underline{A} + \operatorname{ax}((\nabla_{\Delta} \bar{u}) \underline{A}^T - \underline{A} (\nabla_{\Delta} \bar{u})^T)
 \end{aligned} \tag{A.23}$$

If $\underline{A}(x) \in \operatorname{End}(\tau_x)$ then

$$\begin{aligned}
 \operatorname{div}_{\Delta}(f \underline{A}) &= f \operatorname{div}_{\Delta} \underline{A} + \underline{A} \nabla_{\Delta} f \\
 \operatorname{div}_{\Delta}(\underline{A}^T \underline{p} \bar{u}) &= (\operatorname{div}_{\Delta} \underline{A}) \cdot \bar{u} + \underline{A} \cdot D \bar{u} \\
 \operatorname{div}_{\Delta}(\bar{u} \times \underline{A}) &= \bar{u} \times \operatorname{div}_{\Delta} \underline{A} + \operatorname{ax}((\nabla_{\Delta} \bar{u}) \underline{A}^T \underline{p} - \underline{p} \underline{A} (\nabla_{\Delta} \bar{u})^T) \\
 (\operatorname{div}_{\Delta} \underline{A}) \cdot \bar{n} &= \underline{A} \cdot \underline{L}
 \end{aligned} \tag{A.24}$$

where $\bar{n}$ is the orientation on S . Furthermore we have

$$\operatorname{div}_{\Delta} \underline{1}_T = \operatorname{div}_{\Delta} \underline{1} = 2M \bar{n} \tag{A.25}$$

See Bowen & Wang [22] for assistance with the proof of these relations.

Let S be a smooth surface with orientation $\bar{n}$. Let C be a smooth curve on S . Along with the Darboux trihedral $(\bar{e} \ \bar{p} \ \bar{b})$ on C (cf. Appendix A.5) we may consider the dextral trihedral $(\bar{e} \ \bar{n} \ \bar{v})$ where $\bar{v} \equiv \bar{e} \times \bar{n}$. Here $\bar{n}$ is considered as a function of s , the arc-length coordinate on C . Derivatives are obtained according to the formula:

$$\frac{d\bar{a}}{ds}(s) = \bar{\omega}(s) \times \bar{a}(s), \quad \bar{a} = \bar{e}, \bar{n}, \bar{v} \tag{A.26}$$

$$\bar{\omega}(s) = \bar{e}(s) \tau_g(s) + \bar{n}(s) \kappa_g(s) + \bar{v}(s) \kappa_n(s)$$

where (κ is the curvature and τ the torsion of C).

$$\tau_g \equiv \tau + \frac{d\varphi}{ds}, \quad \kappa_g \equiv \kappa \sin \varphi, \quad \kappa_n \equiv \kappa \cos \varphi \quad (\text{A.27})$$

are the geodesic torsion, the geodesic curvature and the normal curvature of C respectively and $\varphi = \varphi(s) \equiv \text{angle}(\bar{p}(s), \bar{n}(s))$. See figure below!

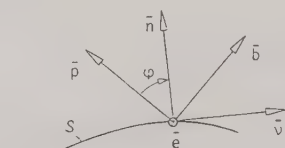


Figure A.1

A moving surface-with-boundary in E is a family of smooth surfaces (-with-boundary) S_t , $t \in T$

$$S_t \ni x = \hat{x}(q_1, q_2, t), \quad (q_1, q_2) \in U \quad (\text{A.28})$$

where we assume that the mapping $\hat{x}(q_1, q_2, \cdot) : T \rightarrow E$ is smooth.

The normal velocity $\bar{v}_t$ of S_t is defined by

$$\bar{v}_t \equiv \bar{n}_t \left(\bar{n}_t \cdot \frac{\partial \hat{x}}{\partial t} \right) \quad (\text{A.29})$$

where $\bar{n}_t = \bar{n}_t(x)$ is an orientation on S_t . This definition is independent of the particular parametrization $(U, \hat{x})$ and orientation $\bar{n}_t$.

The normal speed v_t of S_t (relative to the orientation $\bar{n}_t$) is defined by

$$v_t \equiv \bar{n}_t \cdot \frac{\partial \hat{x}}{\partial t} \quad (\text{A.30})$$

so that $\bar{v}_t = \bar{n}_t v_t$. If (locally) we have $S_t = \{x \in E \mid f(x, t) = 0\}$ for some smooth $f : E \times T \rightarrow \mathbb{R}$, then

$$v_t = - \frac{1}{|\nabla f|} \frac{\partial f}{\partial t}, \quad \bar{n}_t = \frac{1}{|\nabla f|} \nabla f \quad (\text{A.31})$$

The boundary ∂S_t of S_t is a moving curve in E . Let $\bar{c}_t$ denote its normal-velocity and let $\bar{v}_t = \bar{v}_t(x) \in T_x(S_t)$ denote the outward pointing unit normal to ∂S_t . Then we may write

$$\bar{c}_t = \bar{n}_t v_t + \bar{v}_t c_t, \quad x \in \partial S_t \quad (\text{A.32})$$

and for a moving surface-with-boundary having zero normal-velocity we have

$$\bar{c}_t = \bar{v}_t c_t \quad (\text{A.33})$$

A.7 Hadamard's lemma

Let $\mathcal{D}$ be a domain in E with smooth boundary-surface $\partial \mathcal{D}$. Let $\Phi: \mathcal{D} \rightarrow \mathbb{R}$ be continuously differentiable. Let Φ and $\nabla \Phi$ approach finite limits $(\Phi)^+$ and $(\nabla \Phi)^+$ as $\partial \mathcal{D}$ is approached upon paths interior to $\mathcal{D}$ (cf. the construction in chapter 1). If $(\Phi)^+: S \rightarrow \mathbb{R}$ is differentiable then

$$\nabla_{\delta}(\Phi)^+(x) \bar{u} = (\nabla \Phi)^+(x) \bar{u}, \quad \bar{u} \in T_x \quad (\text{A.34})$$

$x \in \partial \mathcal{D}$ (T_x is the tangent space of $\partial \mathcal{D}$ at x).

Proof: See Truesdell & Toupin [5], section 174. $\square$

A.8 The divergence theorem

Let $\mathcal{D}$ be a bounded domain (open, connected) in E . In the following description of its boundary $\partial \mathcal{D}$ we assume that for every point $x \in \partial \mathcal{D}$ there exist numbers $r > 0$, $\delta > 0$ and $\varepsilon > 0$ (depending on x) and a rectangular Cartesian coordinate system $(\bar{e}_1 \bar{e}_2 \bar{e}_3)_o$ with the following properties: If we denote by $\bar{B}(x, r)$ the closed ball in E centered at x and with radius r , by Γ the set (surface) $\partial \mathcal{D} \cap \bar{B}$ and by Δ the set $\{(q_1, q_2) \in \mathbb{R}^2 \mid -\delta \leq q_i \leq \delta, i = 1, 2\}$, then

- (i) there is a function $b: \Delta \rightarrow \mathbb{R}$ satisfying the Lipschitz condition:
There exists a constant $C > 0$ such that for all $(q_1, q_2), (q'_1, q'_2) \in \Delta$ the inequality

$$|b(q_1, q_2) - b(q'_1, q'_2)| \leq C \sqrt{(q_1 - q'_1)^2 + (q_2 - q'_2)^2} \quad (\text{A.35})$$

is satisfied.

$$(ii) \quad \Gamma = \{x \in E \mid x = o + \bar{e}_1 q_1 + \bar{e}_2 q_2 + \bar{e}_3 q_3, \quad q_3 = b(q_1, q_2), \\ (q_1, q_2) \in \Delta\}$$

(iii) the set

$$U^- = \{x \in E \mid x = o + \bar{e}_1 q_1 + \bar{e}_2 q_2 + \bar{e}_3 q_3, \quad b(q_1, q_2) - \varepsilon < q_3 < \\ b(q_1, q_2), \quad (q_1, q_2) \in \Delta\}$$

is a subset of $\mathcal{D}$.

(iv) the set

$$U^+ = \{x \in E \mid x = o + \bar{e}_1 q_1 + \bar{e}_2 q_2 + \bar{e}_3 q_3, \quad b(q_1, q_2) < q_3 < \\ b(q_1, q_2) + \varepsilon, \quad (q_1, q_2) \in \Delta\}$$

is a subset of $E \setminus \bar{\mathcal{D}}$.

A domain $\mathcal{D}$ of the type described above is called a regular region in E .

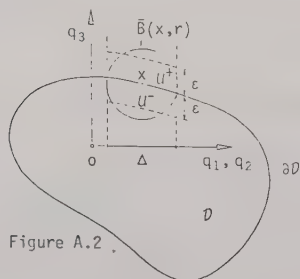


Figure A.2

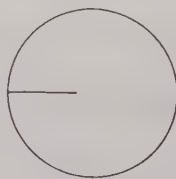


Figure A.3

Remark: Note that the region representing a fracturing body (in the referential description, see Figure A.3 above) is not regular.

We may define an outward pointing unit normal vector $\bar{n}$ (almost everywhere) on Γ by

$$\bar{n} = \frac{1}{\beta} (\bar{e}_1 (-\frac{\partial b}{\partial q_1}) + \bar{e}_2 (-\frac{\partial b}{\partial q_2}) + \bar{e}_3) \quad (A.36)$$

$$\beta = (1 + (\frac{\partial b}{\partial q_1})^2 + (\frac{\partial b}{\partial q_2})^2)^{\frac{1}{2}}$$

Remark: It may be proved that b is differentiable almost everywhere and

that $\frac{\partial b}{\partial q_i} \in L^\infty(\Delta)$. See Necăs [25].

For a function $f: \Gamma \rightarrow \mathbb{R}$ we define the surface integral:

$$\int_{\Gamma} f(x) da(x) \equiv \int_{\Delta} (f \circ q)(q_1, q_2) \beta(q_1, q_2) dq_1 dq_2 \quad (A.37)$$

where the mapping $q: \Delta \rightarrow \Gamma$ is defined by

$$q(q_1, q_2) \equiv o + \bar{e}_1 q_1 + \bar{e}_2 q_2 + \bar{e}_3 b(q_1, q_2), \quad (q_1, q_2) \in \Delta$$

The integral

$$\int_{\partial \mathcal{D}} f(x) da(x)$$

can be defined by a technique involving (A.37) and the partition of unity. See Necăs [25].

Remark: For a vector-valued function $\bar{u}: \Gamma \rightarrow V$ we write

$$\int_{\Gamma} \bar{u}(x) da(x) \equiv \sum_{i=1}^3 \bar{e}_i \left(\int_{\Gamma} u_i(x) da(x) \right)$$

The divergence theorem in E : Let $\mathcal{D}$ be a regular region in E . Let $K: \bar{\mathcal{D}} \rightarrow V$, $\text{End}(V)$ be a vector- or tensor-field on $\mathcal{D}$ satisfying:

(i) K is continuous.

(ii) $\text{div} K \in L^p(\mathcal{D})$, $p > 1$

Then

$$\int_{\partial \mathcal{D}} K \bar{n} da(x) = \int_{\mathcal{D}} \text{div} K da(x) \quad (A.38)$$

Proof: See Necăs [25], ("Formule de Green"). $\square$

Let S be a smooth surface in E with orientation $\bar{n}$. Analogous to the previous discussion we define a regular region on S as a bounded domain d on S whose boundary ∂d is (locally) describable by the means of a function satisfying the Lipschitz condition and the domain d "lies on one side of the boundary ∂d ".

We may introduce the outward unit normal on $\partial d: \bar{v} = \bar{v}(x) \in T_x(S)$ (almost

everywhere $x \in \partial \mathcal{D}$) and the line integral:

$$\int_{\partial d} f(x) ds(x)$$

The divergence theorem on S : Let S be a smooth surface in E with orientation $\bar{n}$ and mean-curvature M . Let d be a regular region on S . Let $k: d \rightarrow V$, $\text{End}(T_x)$, $L(T_x, V)$, $\text{End}(V)$ be a vector- or tensor-field on d satisfying:

(i) k is continuous.

(ii) $\text{div}_d k \in L^p(d)$, $p > 1$

Then

$$\int_{\partial d} k \bar{v} ds(x) = \int_d (\text{div}_d k + 2Mk\bar{n}) da(x) \quad (\text{A.39})$$

Remark: If $k(x) \in \text{End}(T_x)$, $L(T_x, V)$ then we put $k(x)\bar{n}(x) = \bar{0}$.

Proof: The proof of this theorem may be based on Necăs [25] ("Formule de Green" in two dimensions). See also Brand [26], chapter VI. $\square$

A.9 The transport theorem

Consider a family of regular regions $\mathcal{D}_t$, $t \in T$ in E (cf. Appendix A.8) and the function $b: \Delta \times T \rightarrow \mathbb{R}$ where $b(\cdot, t): \Delta \rightarrow \mathbb{R}$ is a Lipschitz-function (related to a coordinate-system $(\bar{e}_1 \bar{e}_2 \bar{e}_3)_0$) describing (locally) the boundary $\partial \mathcal{D}_t$. If the function $b(q_1, q_2, \cdot): T \rightarrow \mathbb{R}$ is smooth then we call $\mathcal{D}_t$, $t \in T$ a moving regular region in E . The normal speed v_t (relative to the orientation (A.36)) is given by

$$v_t = \frac{1}{\beta} \frac{\partial b}{\partial t} \quad (\text{A.40})$$

The transport theorem in E : Let $\mathcal{D}_t$, $t \in T$ be a moving regular region in E with normal velocity v_t . Put

$$\mathbb{D} = \bigcup_{t \in T} (\mathcal{D}_t, t) \quad (\text{A.41})$$

an open set in $E \times \mathbb{R}$.

Consider the scalar- or vector-valued field $\phi: \mathbb{D} \rightarrow \mathbb{R}$, V . If ϕ and $\frac{\partial \phi}{\partial t}$

are continuous on $\bar{D}$ then

$$\frac{d}{dt} \int_{D_t} \Phi(x, t) dv(x) = \int_{D_t} \frac{\partial \Phi}{\partial t}(x, t) dv(x) + \int_{\partial D_t} \Phi(x, t) v_t(x) da(x) \quad (A.42)$$

Proof: Omitted. A proof may be based on Stokes' theorem in $\mathbb{R}^4$. $\square$

Let S be a smooth surface in E . Analogous to the previous discussion we define a moving regular region on S as a family d_t , $t \in T$ of regular regions on S whose boundaries ∂d_t are (locally) describable by a Lipschitz-function which is smooth in the time-parameter.

The transport theorem on S : Let S be a smooth surface in E . Let d_t , $t \in T$ be a moving regular region on S with (outward) normal speed c_t . Put

$$\bar{d} = \bigcup_{t \in T} (d_t, t) \quad (A.43)$$

a surface in $E \times \mathbb{R}$.

Consider a scalar- or vector-valued field $\Psi: \bar{d} \rightarrow \mathbb{R}, V$. If Ψ and $\frac{\partial \Psi}{\partial t}$ are continuous on $\bar{d}$ then

$$\frac{d}{dt} \int_{d_t} \Psi(x, t) da(x) = \int_{d_t} \frac{\partial \Psi}{\partial t}(x, t) da(x) + \int_{\partial d_t} \Psi(x, t) c_t(x) ds(x) \quad (A.44)$$

Proof: Omitted. $\square$

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PAPER II

CRACK PROPAGATION IN A HYPERELASTIC MATERIAL

AND

THE RICE FRACTURE CRITERION

by

Per Lidström

ABSTRACT

In the present paper we consider a two-dimensional body of hyperelastic material. The body is assumed to contain a straight-lined edge-crack. The model takes into account the possibility of non-zero surface tensions on the crack-surfaces.

A boundary-initial-value problem for the fracturing body is formulated. We give special attention to the equation representing the energy-balance of the fracturing body. By introducing kinematical regularity assumptions, in the main based on a convected time derivative associated with the motion of the crack-tip, the equation representing the energy-balance may be given a form previously announced by J.R. Rice.

We also present a formula for the determination of the crack-tip speed in materials that are stable in the sense of Drucker.

1. INTRODUCTION

In a paper entitled: An examination of the Fracture Mechanics Energy Balance from the Point of view of Continuum Mechanics, J.R. Rice derived a so-called fracture criterion for an elastic (elastic-perfectly plastic) solid (see Rice [1], formula (8)). The criterion is based on energy considerations and, with notation as in [1], it is written

$$\Gamma = \lim_{A' \rightarrow 0} \frac{1}{A'} \int_{V_0} \left(\int_{(a)}^{(b)} ((\sigma_{ij}^b - \sigma_{ij}^a) d\epsilon_{ij} + \rho (\ddot{u}_i^b - \ddot{u}_i^a) du_i) \right) dV \quad (1.1)$$

where Γ is the surface energy (interpreted as the work per unit area required to create new surface), ρ is the mass density, u_i the displacement, $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ the strains and σ_{ij} the stresses (infinitesimal deformation gradients are assumed). V_0 is any finite region surrounding the crack-tip. It is assumed that the crack extends, under constant surface traction T_i on A_T and surface displacement U_i on A_U , between two states (a) and (b) so that the traction-free crack-surface increases by an amount A' . See Figure 1.1 below. (F_i = the body force)

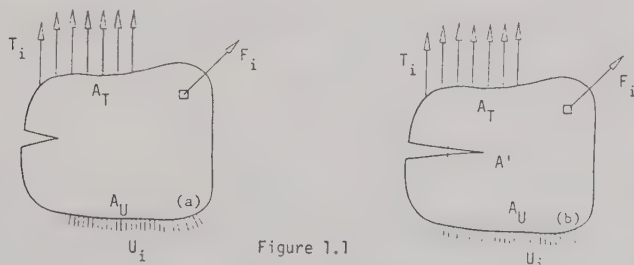


Figure 1.1

The discussion in Rice [1] is mainly heuristic. The kinematical conditions are not specified in detail and a formal application of the divergence theorem, appearing in the proof of (1.1), may be questioned (see the discussions in Luxmoore & Morgan [2] and Gradin [3]). Moreover the assumption that the surface energy Γ is a constant, independent of the present configuration of the body, is an unnecessary primitive comprehension of this concept.

In this paper we state, in section 2, a boundary-initial-value problem for a two-dimensional elastic body under iso-thermal conditions, containing a

¹ rectangular Cartesian coordinates, $u_{i,j} = \partial u_i / \partial x_j$

straight-lined crack (in the reference configuration). The model is more general than the one used in [1] in that we allow for non-zero surface tensions on the crack-surfaces.

In section 3 we prove a formula similar to the Rice fracture criterion (1.1) (see (VI)' in Theorem 1 and (VI)'_a in Theorem 1a). The formula is obtained at the price of rather severe kinematical assumptions.

In section 4 we discuss quasi-static conditions, that is, kinematical assumption leading up to the conclusion

$$\lim_{A' \rightarrow 0} \frac{1}{A'} \int_{V_0}^{(b)} \rho (\ddot{u}_i^b - \ddot{u}_i) du_i dV = 0 \quad (1.2)$$

We also discuss a stability concept, introduced by Drucker (see Drucker [4]), and its implications on the propagation of the crack (section 5).

Our analysis is based on the papers Gurtin & Yatomi [5] and Lidström [6] where several of the basic ideas used in this text were first presented.

Notation: The analysis takes place in a two-dimensional Euclidean space $\mathbb{R}^2$. Points in $\mathbb{R}^2$ are denoted: $x, y, \dots$, vectors: $\bar{a}, \bar{b}, \dots$, and tensors ("linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ ") $\underline{A}, \underline{B}, \dots$; $\underline{1}$ is the unit tensor, $\underline{A}^T$ is the transpose of $\underline{A}$ and $\det \underline{A}$ is the determinant of $\underline{A}$. In standard indicial notation and rectangular Cartesian coordinates we have (the usual summation convention is employed)

$$\bar{a} \cdot \bar{b} = a_i b_i, \quad \underline{A} \cdot \underline{B} = A_{ij} B_{ij}$$

$$\text{div } \underline{T} \text{ is a vector with components } \partial T_{ij} / \partial x_j$$

$$\nabla \bar{u} \text{ is a tensor with components } \partial u_i / \partial x_j$$

$$\partial W / \partial \underline{H} \text{ is a tensor with components } \partial W / \partial H_{ij}$$

$$\partial^2 W / \partial \underline{H}^2 \text{ is a (4th order) tensor with components } \partial^2 W / \partial H_{ij} \partial H_{kl}$$

$$\text{End}(\mathbb{R}^2) \equiv \{ \text{tensors in } \mathbb{R}^2 \}$$

$$\text{HEnd}(\mathbb{R}^2) \equiv \{ \underline{H} \in \text{End}(\mathbb{R}^2) \mid \det(\underline{1} + \underline{H}) > 0 \}$$

$$O(\mathbb{R}^2) \equiv \{ \underline{Q} \in \text{End}(\mathbb{R}^2) \mid \underline{Q} \underline{Q}^T = \underline{Q}^T \underline{Q} = \underline{1} \}$$

$$C^m(B) \equiv \{ m\text{-continuously differentiable functions on } B \}$$

da denotes (Lebesgue) area-measure and ds denotes arc-length measure.

$$L^p(B) \equiv \{ p\text{-integrable functions on } B \}$$

2. BASIC EQUATIONS

A two-dimensional body is identified with a reference configuration B , a bounded, regular region in $\mathbb{R}^2$. Let $\bar{n}$ denote the outward unit normal to ∂B . We consider a fixed time-interval $[0, T]$. We assume that B contains a straight-lined edge-crack C_t extending in its line.

Choose a rectangular Cartesian coordinate system: $(\bar{e}_1, \bar{e}_2)$, $x = (x_1, x_2)$ in $\mathbb{R}^2$ so that

$$C_t = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \ell(t), x_2 = 0\}$$

$t \in [0, T]$, where the function

$$\ell : [0, T] \rightarrow \mathbb{R}^+ \quad (2.1)$$

represents the length of the crack.

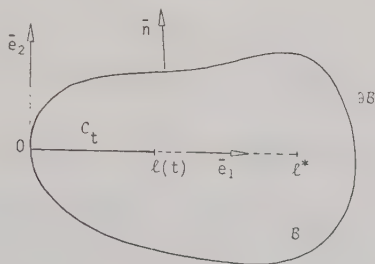


Figure 2.1

Put

$$\ell^* = \sup_{t \in [0, T]} \ell(t) \quad (2.2)$$

$$C^* = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \ell^*, x_2 = 0\} \quad (2.3)$$

Let

$$\bar{u}(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}^2 \quad (2.4)$$

denote the displacement of the body at time $t \in [0, T]$. We assume that for each t the mapping

$$B \setminus C_t \ni x \leadsto x + \bar{u}(x, t)$$

is smooth and one-to-one onto its image and with smooth inverse, and that

$$\det(\underline{1} + \nabla \bar{u}(x, t)) > 0, \quad x \in B \setminus C_t, \quad t \in [0, T]$$

The displacement of the crack at time t is given by

$$\bar{u}^+(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}^2; \quad \bar{u}^-(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}^2 \quad (2.5)$$

where

$$\begin{aligned} \bar{u}^+(q, t) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 > 0}} \bar{u}(x, t), & \bar{u}^-(q, t) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 < 0}} \bar{u}(x, t), & q &\in [0, \ell(t)[\end{aligned} \quad (2.6)$$

$$\begin{aligned} \bar{u}^+(\ell(t), t) &= \bar{u}^-(\ell(t), t) \equiv \lim_{\substack{x \rightarrow (\ell(t), 0) \\ x \in B \setminus C_t}} \bar{u}(x, t) \end{aligned} \quad (2.7)$$

We assume that for each t the mappings:

$$[0, \ell(t)] \ni q \leadsto (q, 0) + \bar{u}^\pm(q, t)$$

are smooth and one-to-one onto their images:

$$C_t^\pm \equiv \{x \in \mathbb{R}^2 \mid x = (q, 0) + \bar{u}^\pm(q, t), \quad q \in [0, \ell(t)]\} \quad (2.8)$$

the "crack-surfaces"¹ in the deformed configuration at time t .

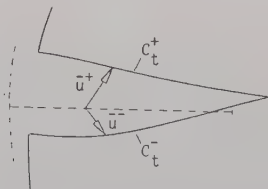


Figure 2.2

¹ "crack-curves" would perhaps be more adequate in our 2-dim. context

Unit tangent vectors on $C_t^\pm$ are given by

$$\bar{e}^\pm(q,t) \equiv \frac{1}{L^\pm(q,t)} (\bar{e}_1 + \frac{\partial \bar{u}^\pm}{\partial q}(q,t)) \quad (2.9)$$

where

$$L^\pm(q,t) \equiv (1 + 2\bar{e}_1 \cdot \frac{\partial \bar{u}^\pm}{\partial q}(q,t) + (\frac{\partial \bar{u}^\pm}{\partial q} \cdot \frac{\partial \bar{u}^\pm}{\partial q})(q,t))^{\frac{1}{2}} \quad (2.10)$$

denote the length-ratios associated with the deformation of the crack. Let

$$\rho : B \rightarrow \mathbb{R}^+ \quad (2.11)$$

denote the continuous reference mass density. Let

$$\mathbb{I}(\cdot, t) : B \setminus C_t \rightarrow \text{End}(\mathbb{R}^2) \quad (2.12)$$

denote the "bulk" (Piola-Kirchhoff) stress tensor and let

$$S^\pm(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R} \quad (2.13)$$

denote the "crack-surface" tensions. The traction field $\bar{S}^\pm = \bar{S}^\pm(\cdot, t)$ on $C_t^\pm$ is then given by

$$\bar{S}^\pm \equiv \bar{e}^\pm S^\pm \quad (2.14)$$

The material under consideration is assumed to be hyperelastic¹ with smooth bulk elastic potential (energy per unit mass)

$$W : \text{HEnd}(\mathbb{R}^2) \times B \rightarrow \mathbb{R} \quad (2.15)$$

$W = W(H, x)$, and smooth "surface" elastic potential (energy per unit crack-length in the reference configuration)

$$\Sigma : \mathbb{R}^+ \times [0, \ell^*] \rightarrow \mathbb{R} \quad (2.16)$$

$\Sigma = \Sigma(L, q)$. We assume that the bulk elastic potential is indifferent under

¹ See Truesdell & Noll [7].

a change of frame, i.e.

$$W(QH + Q - I, x) = W(H, x) \quad (2.17)$$

for all $H \in \text{HEnd}(\mathbb{R}^2)$, $Q \in O(\mathbb{R}^2)$. Note that this will imply indifference of bulk stress and local balance of moment of momentum at a bulk material point (i.e. symmetry of the Cauchy stress tensor. See Truesdell & Noll [7], section 84!)

Denote by $\mathcal{D}$ the set of displacement fields $\bar{u}$ satisfying the items (i) - (x) stated below:

(i) $\ell \in C^1([0, T])$, $\ell(t) > 0$ for $t \in [0, T]$ and $C^* \subset \mathring{B}$.

From this assumption it follows that

(a) for any time $t \in]0, T[$ and any $x \in B \setminus C_t$ there exists $\varepsilon > 0$ and an open disc $D(x, \delta)$ such that $D \cap C_s = \emptyset$ for all $s \in]t - \varepsilon, t + \varepsilon[$.

(b) for any time $t \in]0, T[$ and any $q \in]0, \ell(t)[$ there exists $\varepsilon > 0$ and $\delta > 0$ such that $]q - \delta, q + \delta[\subset]0, \ell(t)[$ for all $s \in]t - \varepsilon, t + \varepsilon[$.

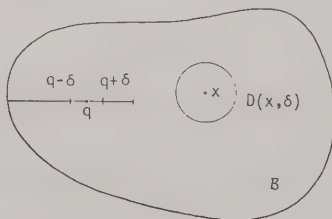


Figure 2.3

(ii) $\bar{u}(\cdot, t) \in C^2(B \setminus C_t)$

(iii) Limits $(\nabla \bar{u})^+$, $(\nabla \bar{u})^{-1}$, in the sense of (2.6), exist for $q \in]0, \ell(t)[$

(iv) If P is an open subset of B and I an open subinterval in $[0, T]$ satisfying $P \cap C_t = \emptyset$ for every $t \in I$ then the mapping

$$P \times I \ni (x, t) \mapsto \bar{u}(x, t)$$

is C^2 .

¹ $\nabla = \partial/\partial x$

From (a) and (iv) it follows that we may introduce bulk material velocity and acceleration

$$\dot{\bar{u}}(x,t) \equiv \partial_t \bar{u}(x,t), \quad \ddot{\bar{u}}(x,t) \equiv \partial_t^2 \bar{u}(x,t), \quad x \in B \setminus C_t \quad (2.18)$$

(v) Limits $(\dot{\bar{u}})^+$, $(\ddot{\bar{u}})^-$, in the sense of (2.6), exist for $q \in [0, \ell(t)[$.

(vi) $\bar{u}^\pm(\cdot, t) \in C^2([0, \ell(t)])$

(vii) If I is an open subinterval of $[0, T]$ and J an open interval in $\mathbb{R}$ satisfying $J \subset [0, \ell(t)[$ for every $t \in I$ then the mappings

$$J \times I \ni (q, t) \leadsto \bar{u}^\pm(q, t)$$

are C^2 .

From (b) and (vii) it follows that we may introduce surface material velocity

$$\dot{\bar{u}}^\pm(q, t) \equiv \partial_t \bar{u}^\pm(q, t), \quad q \in [0, \ell(t)[\quad (2.19)$$

(viii) The fields $\dot{\bar{u}}^\pm(\cdot, t) \cdot \bar{e}^\pm(\cdot, t)$ may be continuously extended to the crack-tip ($q = \ell(t)$).

(ix) The crack-surfaces in the deformed configuration, C_t^+ and C_t^- , are nowhere in contact (except, of course at the crack-tip).

(x) The mappings:

$$[0, T] \ni t \leadsto \int_B \dot{\bar{u}}(x, t) \rho(x) da(x)$$

$$[0, T] \ni t \leadsto \int_B (\chi(x, t) - x_0) \times \dot{\bar{u}}(x, t) \rho(x) da(x)^2$$

$$[0, T] \ni t \leadsto \int_B \left(\frac{1}{2} \dot{\bar{u}}(x, t) \cdot \dot{\bar{u}}(x, t) + W(\nabla \bar{u}(x, t), x) \right) \rho(x) da(x)$$

where x_0 is a fix point in $\mathbb{R}^2$ and $\chi(x, t) \equiv x + \bar{u}(x, t)$, are C^1 .

Remark 1: Note that the assumptions in (ii) and (iv) do not allow for

¹ $\partial_t = \partial/\partial t$ ² $\times$ denotes the vector-product in $\mathbb{R}^3$

shock waves, acceleration waves etc. in $B \setminus C_t$.

Remark 2: The assumptions made in (x) will guarantee the existence and differentiability of functions representing bulk momentum, moment of momentum and energy of the body. This is a non-trivial assumption since the fields $\bar{u}$ and $\nabla \bar{u}$ may be singular (unbounded) at the crack-tip and thus (x) does not follow from the preceding assumptions (i) - (ix).

The crack-tip speed (in the reference configuration):

$$c : [0, T] \rightarrow \mathbb{R} \quad (2.20)$$

is defined by

$$c(t) = \frac{d\ell}{dt}(t) \quad (2.21)$$

Remark 1: Note that c depends on the choice of reference configuration. See Lidström [6] for a discussion on this dependence.

Remark 2: If $c(t) > 0$ for $t \in [0, T]$ then ℓ is a strictly increasing function of t ("an advancing crack") and we may use $\ell \in [\ell_0, \ell^*]$ ($\ell_0 = \ell(0)$) as an independent variable instead of t .

We may now state a boundary-initial-value problem for the fracturing, hyperelastic body B :

Find $\bar{u} \in \mathcal{D}$ so that

$$\operatorname{div} \bar{I} + \bar{b} \rho = \ddot{\bar{u}} \rho, \quad x \in B \setminus C_t \quad (I) \quad (2.22)$$

$$\bar{I}(x, t) = \rho(x) \frac{\partial W}{\partial H}(\nabla \bar{u}(x, t), t) \quad (2.22)$$

where

the body force

$$\bar{b} : B \times [0, T] \rightarrow \mathbb{R}^2 \quad (2.23)$$

is a prescribed continuous field.

$$\frac{\partial}{\partial q} \bar{S}^+ + (I)^+ \bar{e}_2 = \bar{0}, \quad q \in [0, \ell(t)[\quad (II)_1$$

$$\frac{\partial}{\partial q} \bar{S}^- - (\bar{I})^- \bar{e}_2 = \bar{0} \quad (\text{II})_2$$

$$\bar{S}^\pm(q, t) \equiv \bar{e}^\pm(q, t) \frac{\partial \Sigma}{\partial L}(L^\pm(q, t), q) \quad (2.24)$$

$$\bar{S}^\pm(0, t) = \bar{0} \quad (\text{III})$$

$$\begin{aligned} \bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t) &= \int_{\partial B} \bar{I} \bar{n} \, ds(x) - \int_0^{\ell(t)} ((\bar{I})^+ - (\bar{I})^-) \bar{e}_2 \, dq + \\ &+ \int_B \bar{b} \, \rho \, da(x) - \frac{d}{dt} \int_B \bar{u} \, \rho \, da(x) \end{aligned} \quad (\text{IV})$$

$$\begin{aligned} (\chi^\pm(\ell(t), t) - x_0) \times (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) &= \int_{\partial B} (\chi - x_0) \times \bar{I} \bar{n} \, ds(x) - \\ &- \int_0^{\ell(t)} ((\chi^+ - x_0) \times (\bar{I})^+ - (\chi^- - x_0) \times (\bar{I})^-) \bar{e}_2 \, dq + \int_B (\chi - x_0) \times \bar{b} \, \rho \, da(x) - \\ &- \frac{d}{dt} \int_B (\chi - x_0) \times \bar{u} \, \rho \, da(x) \end{aligned} \quad (\text{V})$$

where $\chi^\pm(q, t) \equiv (q, 0) + \bar{u}^\pm(q, t)$, $q \in [0, \ell(t)[$ and $\chi(x, t) \equiv x + \bar{u}(x, t)$, $x \in B \setminus C_t$ (x_0 is a fix point in $\mathbb{R}^2$).

$$\begin{aligned} (\sigma^+(\ell(t), t) + \sigma^-(\ell(t), t)) c(t) + \dot{\bar{u}}^+(\ell(t), t) \cdot \bar{S}^+(\ell(t), t) + \\ \dot{\bar{u}}^-(\ell(t), t) \cdot \bar{S}^-(\ell(t), t) &= \int_{\partial B} \dot{\bar{u}} \cdot \bar{I} \bar{n} \, ds(x) - \int_0^{\ell(t)} (\dot{\bar{u}}^+ \cdot (\bar{I})^+ \bar{e}_2 - \\ &- \dot{\bar{u}}^- \cdot (\bar{I})^- \bar{e}_2) \, dq + \int_B \dot{\bar{u}} \cdot \bar{b} \, \rho \, da(x) - \frac{d}{dt} \int_B \left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \rho \, da(x) \end{aligned} \quad (\text{VI})$$

where the bulk strain-energy density w and the surface strain-energy densities $\sigma^\pm$ are defined by

$$w(x, t) \equiv W(\nabla \bar{u}(x, t), x), \quad \sigma^\pm(q, t) \equiv \Sigma(L^\pm(q, t), q) \quad (2.25)$$

$$\bar{I} \bar{n} = \bar{t}, \quad x \in \partial B \quad (\text{VII})$$

where

the boundary traction

$$\bar{t} : \partial B \times [0, T] \rightarrow \mathbb{R}^2 \quad (2.26)$$

is a prescribed continuous field.

$$\ell(0) = \ell_0$$

$$\bar{u}(x, 0) = \bar{u}_0(x) \quad (VIII)$$

$$\dot{\bar{u}}(x, 0) = \bar{v}_0(x) \quad x \in B \setminus C_{t=0}$$

where

$\ell_0 > 0$ and the initial-values $\bar{u}_0, \bar{v}_0 : B \setminus C_{t=0} \rightarrow \mathbb{R}^2$ are continuous with limits $(\bar{u}_0)^\pm, (\bar{v}_0)^\pm$ in the sense of (2.6).

Remark 1: It may be demonstrated that equations (I) - (VII) are equivalent to the global balance of momentum, moment of momentum and energy for every subbody of B when B is undergoing a displacement $\bar{u} \in \mathcal{D}$. In the proof of this equivalence we actually need some additional regularity assumptions on $\bar{u}$ not presented here. See Lidström [6] where these matters are discussed at length.

Remark 2: Note that (2.17) implies, not only the frame indifference of bulk stress, but also the relation

$$(I + \nabla \bar{u}) I^T = I (I + \nabla \bar{u}^T) \quad (2.27)$$

expressing the local balance of moment of momentum at a bulk material point. Furthermore, from (2.22), (2.24) and (2.25), it follows that

$$\dot{w} = \frac{1}{\rho} I \cdot \nabla \dot{\bar{u}}, \quad \dot{\sigma}^\pm = \bar{S}^\pm \cdot \frac{\partial \dot{\bar{u}}^\pm}{\partial q} \quad (2.28)$$

Remark 3: Let $\bar{u}$ be a solution to the boundary-initial-value problem. The right membrum of equation (IV) may be called the momentum release rate for B at time t :

$$M(B,t) = \int_{\partial B} \underline{I} \bar{n} ds(x) - \int_0^{\ell(t)} ((\underline{I})^+ - (\underline{I})^-) \bar{e}_2 dq + \int_B \bar{b} \rho da(x) - \frac{d}{dt} \int_B \dot{\bar{u}} \rho da(x) \quad (2.29)$$

If P is a subbody (a regular subregion) of B containing the crack-tip at time t then put

$$M(P,t) = \int_{\partial P} \underline{I} \bar{n} ds(x) - \int_{\ell_P}^{\ell(t)} ((\underline{I})^+ - (\underline{I})^-) \bar{e}_2 dq + \int_P \bar{b} \rho da(x) - \frac{d}{dt} \int_P \dot{\bar{u}} \rho da(x) \quad (2.30)$$

We assume that ∂P intersects C_t at only one point $(\ell_P, 0)$.

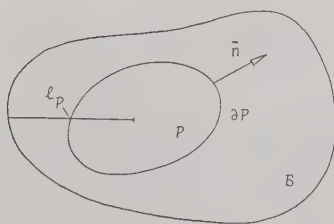


Figure 2.4

From the divergence theorem and (I) it follows that

$$M(P,t) = M(B,t) \quad (2.31)$$

that is, the definition of the momentum release rate is independent of the subbody surrounding the crack-tip.

Analogously the right membra of equations (V) and (VI) may be called the moment of momentum release rate $mM(B,t)$ and the energy release rate $E(B,t)$ respectively. By invoking (I), (2.27) and (2.28)₁ (and the divergence theorem) we may show that

$$mM(P,t) = mM(B,t), \quad E(P,t) = E(B,t) \quad (2.32)$$

for every subbody P containing the crack-tip at time t .

Note that if $\bar{u} \in \mathcal{D}$ is sufficiently regular then

$$\int_{\partial B} \bar{\mathbb{I}} \bar{n} ds(x) - \int_0^{t(t)} ((\bar{\mathbb{I}})^+ - (\bar{\mathbb{I}})^-) \bar{e}_2 dq = \int_B \operatorname{div} \bar{\mathbb{I}} da(x) \quad (2.33)$$

and

$$\frac{d}{dt} \int_B \dot{\bar{u}} \rho da(x) = \int_B \ddot{\bar{u}} \rho da(x) \quad (2.34)$$

and we may conclude, from (I), that

$$M(B, t) = \bar{0} \quad (2.35)$$

Conditions (on $\bar{\mathbb{I}}$ and $\bar{u}$) guaranteeing the validity of (2.33) and (2.34) are presented in the Appendix (see Lemmas A.2 and A.6). Analogous conclusions may be drawn for mM and E . See also Lidström [6].

The boundary-initial-value problem stated in this section is one of formidable complexity, containing, apart from conditions in the form of nonlinear differential equations: (I), (II), a number of conditions at the crack-tip of a rather intricate nature: (IV) - (VI). An important subproblem in the study of this boundary-initial-value problem is to find simplified and more transparent versions of the crack-tip conditions. This will require the introduction of additional restrictions on the displacement field $\bar{u}$.

In the next section we present a reformulation of the crack-tip conditions based on an idea presented in Rice [1] and subsequently discussed in Gurtin & Yatomi [5].

3. CRACK-TIP CONDITIONS AND THE RICE FRACTURE CONDITION

We begin this section by introducing some basic definitions. We assume that $\ell \in C^1([0, T])$. Let

$$\varphi(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2 \quad (3.1)$$

$t \in [0, T]$ be a family of fields satisfying:

$$(\alpha) \quad \varphi(\cdot, t) \in C^1(B \setminus C_t)$$

(β) If P is an open subset of B and I an open subinterval in $[0, T]$ satisfying $P \cap C_t = \emptyset$ for every $t \in I$ then the mapping

$$P \times I \ni (x, t) \mapsto \varphi(x, t)$$

is C^1 .

We may then introduce the convected crack-tip time-derivative

$$\varphi'(x, t) \equiv \frac{d}{ds} \varphi(x + \bar{e}_1 c(t)s, t + s) \big|_{s=0}, \quad x \in B \setminus C_t \quad (3.2)$$

We have the important relation

$$\dot{\varphi} = \varphi' - \partial_C \varphi \quad (3.3)$$

where $\partial_C \varphi$ is defined by

$$\partial_C \varphi(x, t) \equiv c(t) \frac{\partial \varphi}{\partial x_1}(x, t) \quad (3.4)$$

For higher order derivatives we put

$$\partial_C^2 \varphi \equiv c^2 \frac{\partial^2 \varphi}{\partial x_1^2}, \quad \partial_C^3 \varphi \equiv c^3 \frac{\partial^3 \varphi}{\partial x_1^3} \quad (3.5)$$

If $\ell \in C^2([0, T])$ and if the mapping in (β) is C^2 then it is a simple matter to confirm the following rules:

$$(\varphi')' = (\dot{\varphi})' + \partial_C \dot{\varphi} \quad (3.6)_1$$

$$\begin{aligned}
 (\partial_C \varphi)^* &= \partial_C \dot{\varphi} + \partial_C^* \varphi \\
 (\partial_C \varphi)' &= \partial_C \varphi' + \partial_C^* \varphi
 \end{aligned}
 \tag{3.6}_{2,3}$$

where

$$\partial_C^* \varphi \equiv \dot{c} \frac{\partial \varphi}{\partial x_1} \tag{3.7}$$

If $\ell \in C^3([0, T])$ and if the mapping in (β) is C^3 then by straightforward calculations we get:

$$\begin{aligned}
 \ddot{\varphi} &= \varphi'' - 2 \partial_C \varphi' - \partial_C^* \varphi + \partial_C^2 \varphi \\
 \ddot{\varphi} &= \varphi''' - 3 \partial_C \varphi'' + 3 \partial_C^2 \varphi' - 3 \partial_C^* \varphi' + 3 \partial_C \partial_C^* \varphi - \partial_C^* \varphi - \partial_C^3 \varphi
 \end{aligned}
 \tag{3.8}$$

Remark 1: Definition (3.2) was first given by H.C. Strifors (see [8], formula (8.23)). See also Gurtin & Yatomi [5] and Lidström [6].

Remark 2: Note that if

$$\Omega = \{ \bar{r} \in \mathbb{R}^2 \mid \bar{r} = (\bar{e}_1 \cos \theta + \bar{e}_2 \sin \theta) r, \ 0 < r < r_0, \ -\pi < \theta < \pi \}$$

and

$$\tilde{\varphi}: \Omega \times [0, T] \rightarrow \mathbb{R}, \mathbb{R}^2$$

is defined by

$$\tilde{\varphi}(\bar{r}, t) \equiv \varphi((\ell(t), 0) + \bar{r}, t) \tag{3.9}$$

then

$$\varphi'(x, t) = \frac{\partial \tilde{\varphi}}{\partial t}(x - (\ell(t), 0), t)$$

Remark 3: Note that

$$\nabla \dot{\varphi} = (\nabla \varphi)^*, \quad \nabla(\partial_C \varphi) = \partial_C(\nabla \varphi), \quad \text{and} \quad \nabla \varphi' = (\nabla \varphi)' \tag{3.10}$$

For the displacements of the crack we take

$$(\bar{u}^\pm)' \equiv \dot{\bar{u}}^\pm + c \frac{\partial \bar{u}^\pm}{\partial q}, \quad q \in [0, \ell(t)[\quad (3.11)$$

then since ¹

$$\dot{\bar{u}}^\pm = (\dot{\bar{u}})^\pm, \quad \frac{\partial \bar{u}^\pm}{\partial q} = \left(\frac{\partial \bar{u}}{\partial x_1} \right)^\pm \quad (3.12)$$

we find that

$$(\bar{u}^\pm)' = (\bar{u}')^\pm \quad (3.13)$$

We write

$$\varphi \in L^p(B), \quad 1 \leq p \leq \infty \quad (3.14)$$

if $\varphi(\cdot, t)$ is measurable on B and

$$\int_B |\varphi(x, t)|^p da(x) < \infty \quad (3.15)$$

for every $t \in [0, T]$.

Following Gurtin & Yatomi [5] we write

$$\varphi \in L^p(B) \text{ uniformly in time} \quad (3.16)$$

if $\varphi \in L^p(B)$ and if, given any one-parameter family B_δ , $\delta > 0$ of measurable subsets of B with $\text{area}(B \setminus B_\delta) \rightarrow 0$ as $\delta \rightarrow 0$, we have

$$\int_{B_\delta} |\varphi(x, t)|^p da(x) \rightarrow \int_B |\varphi(x, t)|^p da(x) \quad (3.17)$$

uniformly in $[0, T]$ as $\delta \rightarrow 0$. In the Appendix (Lemma A.1) we present a few elementary results in connection with this definition.

For a family of fields

$$\psi(\cdot, t) : [0, \ell(t)[\rightarrow \mathbb{R}, \mathbb{R}^2 \quad (3.18)$$

$t \in [0, T]$ we write

¹ consequences of the so-called Hadamard's Lemma, see Lidström [6]

$$\psi \in L^p(C) \quad (3.19)$$

if $\psi(\cdot, t)$ is measurable on $[0, \ell(t)[$ and

$$\int_0^{\ell(t)} |\psi(q, t)|^p dq < \infty \quad (3.20)$$

for every $t \in [0, T]$. We write

$$\psi \in L^p(B) \text{ uniformly in time} \quad (3.21)$$

if $\psi \in L^p(B)$ and if, given any one-parameter family $I_{t, \delta}$, $\delta > 0$ of measurable subsets of $[0, \ell(t)[$ with $\text{length}([0, \ell(t)[\setminus I_{t, \delta}) \rightarrow 0$ as $\delta \rightarrow 0$, we have

$$\int_{I_{t, \delta}} |\psi(q, t)|^p dq \rightarrow \int_0^{\ell(t)} |\psi(q, t)|^p dq \quad (3.22)$$

uniformly in $[0, T]$ as $\delta \rightarrow 0$.

With these definitions at our disposal we may now introduce the additional restrictions on the displacement field required for our reformulation of the crack-tip conditions.

Denote by $\mathcal{D}'$ the set of displacement fields $\bar{u} \in \mathcal{D}$ satisfying the items (xi) - (xvi) stated below:

(xi) $\ell \in C^2([0, T])$ and $c(t) \geq 0$ for $t \in [0, T]$

(xii) $\bar{u}$ is bounded in the sense that $\sup_{\substack{x \in B \setminus C_t \\ t \in [0, T]}} |\bar{u}(x, t)| < \infty$

(xiii) $\nabla \bar{u} \in L^2(B)$, $\frac{\partial \bar{u}}{\partial x_1} \in L^2(B)$ uniformly in time and $\frac{\partial^2 \bar{u}}{\partial x_1^2} \in L^1(B)$

(xiv) $\bar{u}' \in L^2(B)$, $\bar{u}''$ and $\frac{\partial \bar{u}'}{\partial x_1} \in L^1(B)$ all uniformly in time

(xv) $\bar{I} \in L^2(B)$ and the integral

$$\int_{O_R(t)} |\bar{I} \bar{e}_R| ds(x)$$

remains bounded as $\delta \rightarrow 0$ where

$$O_R(t) \equiv \{x \in \mathbb{R}^2 \mid (x_1 - \ell(t))^2 + x_2^2 = R^2\}$$

and $\bar{e}_R = \bar{e}_R(t)$ is the outward pointing unit normal on $O_R(t)$.

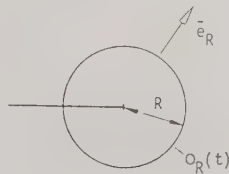


Figure 3.1

(xvi) The mappings

$$\Omega \ni (q, t) \sim \bar{u}^\pm(x, t), \quad \frac{\partial \bar{u}^\pm}{\partial q}(q, t), \quad \frac{\partial^2 \bar{u}^\pm}{\partial q^2}(q, t)$$

where

$$\Omega \equiv \{(q, t) \mid 0 \leq q \leq \ell(t), \quad 0 \leq t \leq T\}$$

are continuous.

(xvii) $(\bar{u}^\pm)' \in L^1(C)$ uniformly in time

Consider the mappings ($\ell^* = \ell(T)$)

$$\bar{u}_*^\pm(\cdot, t) : [0, \ell^*] \rightarrow \mathbb{R}^2$$

where

$$\begin{aligned} \bar{u}_*^+(q, t) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 > 0}} \bar{u}(x, t), & \bar{u}_*^-(q, t) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 < 0}} \bar{u}(x, t), & q &\in [0, \ell^*] \end{aligned} \quad (3.23)$$

i.e. $\bar{u}_*^\pm(\cdot, t)$ are natural extensions of $\bar{u}^\pm(\cdot, t)$ to $[0, \ell^*]$. Note that $\bar{u}_*^+(q, t) = \bar{u}_*^-(q, t) = \bar{u}((q, 0); t)$ for $q \in [\ell(t), \ell^*]$.

We now assume that

$$(xviii) \quad \bar{u}_*^{\pm}(\cdot, t) \in C^1([0, \ell^*]) \quad \text{for } t \in [0, T]$$

Remark: From (xviii) it follows that C_t^+ and C_t^- are tangential at the crack-tip.

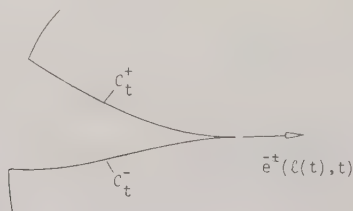


Figure 3.2

$$(xix) \quad \frac{\partial}{\partial q} \dot{\bar{S}}^{\pm} \in L^1(C) \quad \text{uniformly in time.}$$

(xx) The mapping

$$[0, T] \ni t \mapsto \bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)$$

is C^1 .

(xxi) The motion of the crack-tip in the deformed configuration, represented by the mapping

$$[0, T] \ni t \mapsto (\ell(t), 0) + \bar{u}^{\pm}(\ell(t), t)$$

is C^1 and we have the relation

$$\bar{c}(t) \cdot \bar{e}^{\pm}(\ell(t), t) = \dot{\bar{u}}^{\pm}(\ell(t), t) \cdot \bar{e}^{\pm}(\ell(t), t) + c(t) L^{\pm}(\ell(t), t) \quad (3.24)$$

where

$$\bar{c}(t) \equiv \frac{d}{dt}((\ell(t), 0) + \bar{u}^{\pm}(\ell(t), t)) \quad (3.25)$$

is the crack-tip velocity in the deformed configuration.

Remark: If $\dot{\bar{u}}^\pm$ may be continuously extended to the crack-tip, then the relation (3.24) follows directly from the definition (3.25) by a computation of the derivative. In (xxi) we have assumed that formula (3.24) holds true even if $\dot{\bar{u}}^\pm$ are singular (unbounded) at the crack-tip. The reader is reminded of the assumption, made in (viii), that the fields $\dot{\bar{u}}^\pm \cdot \bar{e}^\pm$ may be continuously extended to the crack-tip. See Lidström [6] for a discussion on these matters in the 3-dimensional context.

Our first result is given by

Theorem 1 If the density ρ and the body force $\bar{b}$ are C^1 -functions and $\bar{u} \in \mathcal{D}'$ then the crack-tip conditions (IV) - (VI) are equivalent to, respectively

$$\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t) = \lim_{R \rightarrow 0} \int_{O_R(t)} \bar{I} \bar{e}_R ds(x) \quad (IV)'$$

$$\bar{0} = \lim_{R \rightarrow 0} \int_{O_R(t)} (\chi - \chi^\pm(\ell(t), t)) \times \bar{I} \bar{e}_R ds(x) \quad (V)'$$

and "the Rice fracture criterion"

$$(\sigma^+(\ell(t), t) + \sigma^-(\ell(t), t)) c(t) =$$

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s ((\bar{I}_s - \bar{I}_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda + \rho (\ddot{\bar{u}}_s - \ddot{\bar{u}}_\lambda) \cdot \dot{\bar{u}}_\lambda) d\lambda \right) da(x) \quad (VI)'$$

where $\left(\frac{d}{ds}\right)_t^r$ denotes the right-derivative with respect to s evaluated at $s = t$. For the sake of convenience we write φ_t instead of $\varphi(\cdot, t)$ for a field defined on $B \setminus C_t$.

Proof: For the proof of (IV)' we need

Lemma 1

$$\frac{d}{dt} \int_B \dot{\bar{u}} \rho da(x) = \int_B \ddot{\bar{u}} \rho da(x) \quad (3.26)$$

Proof of Lemma 1: If $\bar{u} \in \mathcal{D}'$ then, from (3.3), (xiii) and (xiv), we find that

$$\dot{\bar{u}} \rho = (\bar{u}' - \partial_{\bar{c}} \bar{u}) \rho \in L^2(B) \text{ uniformly in time} \quad (3.27)$$

$$(\dot{\bar{u}} \rho)' = (\bar{u}'' - \partial_{\bar{c}} \bar{u}' - \partial_{\bar{c}}^2 \bar{u}) \rho + \dot{\bar{u}} \partial_{\bar{c}} \rho \in L^1(B) \text{ uniformly in time} \quad (3.28)$$

$$\partial_{\bar{c}}(\dot{\bar{u}} \rho) = (\partial_{\bar{c}} \bar{u}' - \partial_{\bar{c}}^2 \bar{u}) \rho + \dot{\bar{u}} \partial_{\bar{c}} \rho \in L^1(B) \quad (3.29)$$

and then (3.26) follows from Lemma A.8 (see Appendix!). $\square$

From (I) and Lemma 1 it follows that (IV) is equivalent to

$$\begin{aligned} \bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t) &= \int_{\partial B} \bar{I} \bar{n} ds(x) - \int_0^{\ell(t)} ((\bar{I})^+ - (\bar{I})^-) \bar{e}_2 dq - \\ &\quad \int_B \operatorname{div} \bar{I} da(x) \end{aligned}$$

and by using Lemma A.6 we find that this is equivalent to (IV)'.

For the proof of (V)' we need

Lemma 2

$$\frac{d}{dt} \int_B (\chi - x_0) \times \dot{\bar{u}} \rho da(x) = \int_B (\chi - x_0) \times \ddot{\bar{u}} \rho da(x) \quad (3.30)$$

Proof of Lemma 2: From (xii), (xiv) and (3.27) - (3.29) we find that

$$(\chi - x_0) \times \dot{\bar{u}} \rho = (\bar{u} + \chi - x_0) \times \dot{\bar{u}} \rho \in L^1(B) \text{ uniformly in time}$$

$$((\chi - x_0) \times \dot{\bar{u}} \rho)' = (\bar{u}' + \bar{e}_1 c) \times \dot{\bar{u}} \rho + (\bar{u} + \chi - x_0) \times (\dot{\bar{u}} \rho)' \in L^1(B) \text{ uniformly in time.}$$

$$\partial_{\bar{c}}((\chi - x_0) \times \dot{\bar{u}} \rho) = (\partial_{\bar{c}} \bar{u} + \bar{e}_1 c) \times \dot{\bar{u}} \rho + (\bar{u} + \chi - x_0) \times \partial_{\bar{c}}(\dot{\bar{u}} \rho) \in L^1(B)$$

and then (3.30) follows from Lemma A.8. $\square$

From Lemma 2 and (2.27) it follows that (V) is equivalent to

$$\begin{aligned} (\chi^\pm(\ell(t), t) - x_0) \times (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) &= \int_{\partial B} (\chi - x_0) \times \bar{I} \bar{n} ds(x) - \\ &\quad \int_0^{\ell(t)} ((\chi^+ - x_0) \times (\bar{I})^+ - (\chi^- - x_0) \times (\bar{I})^-) \bar{e}_2 dq - \int_B \operatorname{div}((\chi - x_0) \times \bar{I}) da(x) \end{aligned}$$

and by invoking Lemma A.6 we find that this is equivalent to

$$(\chi^{\pm}(\ell(t), t) - x_0) \times (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) =$$

$$\lim_{R \rightarrow 0} \int_{O_R(t)} (\chi - x_0) \times \bar{I} \bar{e}_R ds(x) \quad (3.31)$$

But from (IV)' we get

$$\lim_{R \rightarrow 0} \int_{O_R(t)} (\chi - x_0) \times \bar{I} \bar{e}_R ds(x) = \lim_{R \rightarrow 0} \int_{O_R(t)} (\chi - \chi^{\pm}(\ell(t), t)) \times \bar{I} \bar{e}_R ds(x) +$$

$$(\chi^{\pm}(\ell(t), t) - x_0) \times (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) \quad (3.32)$$

and then, from (3.31) and (3.32), it follows that (V) is equivalent to (V)'.

We introduce the following simplifying notation: If $\varphi_t: B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$ is a field with limits $(\varphi_t)^+$, $(\varphi_t)^-$, in the sense of (2.6), then the jump of φ_t

$$[\varphi_t]: [0, \ell(t)[\rightarrow \mathbb{R}, \mathbb{R}^2 \quad (3.33)$$

is defined by

$$[\varphi_t](q) \equiv (\varphi_t)^+(q) - (\varphi_t)^-(q) \quad (3.34)$$

The proof of (VI)' is organized in a number of Lemmas:

Lemma 3 For $s \geq t$ we have

$$\begin{aligned} & \int_{\partial B} (\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{n} ds(x) - \int_0^{\ell(s)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{e}_2] dq = \\ & \int_B ((\nabla \bar{u}_s - \nabla \bar{u}_t) \cdot \bar{I}_s + (\bar{u}_s - \bar{u}_t) \cdot (\ddot{\bar{u}}_s - \ddot{\bar{b}}_s) \rho) da(x) + \\ & (\bar{u}^{\pm}(\ell(s), s) - \bar{u}_*^{\pm}(\ell(s), t)) \cdot (\bar{S}^+(\ell(s), s) + \bar{S}^-(\ell(s), s)) \end{aligned} \quad (3.35)$$

Proof of Lemma 3: From (3.8)₁ (with $\varphi = \bar{u}$) we get

$$\ddot{\bar{u}} = \bar{u}'' - 2\partial_C \bar{u}' - \partial_C \bar{u} + \partial_C^2 \bar{u}$$

and thus, from (xiii) and (xiv), we may conclude that

$$\ddot{\bar{u}} \in L^1(B) \quad (3.36)$$

By using (I) we find that

$$\operatorname{div}(\mathbb{I}_S^T(\bar{u}_S - \bar{u}_t)) = (\ddot{\bar{u}}_S - \bar{b}_S) \rho \cdot (\bar{u}_S - \bar{u}_t) + \mathbb{I}_S \cdot (\nabla \bar{u}_S - \nabla \bar{u}_t)$$

and thus, from (3.36), (xii), (xiii) and (xv), it follows that

$$\operatorname{div}(\mathbb{I}_S^T(\bar{u}_S - \bar{u}_t)) \in L^1(B)$$

and then, by invoking Lemma A.6, we have (for $s \geq t$)

$$\int_{\partial B} (\bar{u}_S - \bar{u}_t) \cdot \mathbb{I}_S \bar{n} \, ds(x) - \int_0^{\ell(s)} [(\bar{u}_S - \bar{u}_t) \cdot \mathbb{I}_S \bar{e}_2] \, dq =$$

$$\int_B ((\ddot{\bar{u}}_S - \bar{b}_S) \rho \cdot (\bar{u}_S - \bar{u}_t) + \mathbb{I}_S \cdot (\nabla \bar{u}_S - \nabla \bar{u}_t)) \, da(x) +$$

$$\lim_{R \rightarrow 0} \int_{O_R(s)} (\bar{u}_S - \bar{u}_t) \cdot \mathbb{I}_S \bar{e}_R \, ds(x)$$

Note that from (xi) it follows that $\ell(s) \geq \ell(t)$ since $s \geq t$.

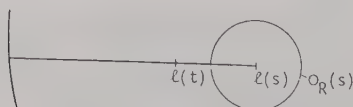


Figure 3.3

From (xv) and (IV)¹ it follows that ($s \geq t$)

$$\lim_{R \rightarrow 0} \int_{O_R(s)} (\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{e}_R ds(x) =$$

$$(\bar{u}^\pm(\ell(s), s) - \bar{u}_*^\pm(\ell(s), t)) \cdot \lim_{R \rightarrow 0} \int_{O_R(s)} \bar{I}_s \bar{e}_R ds(x) =$$

$$(\bar{u}^\pm(\ell(s), s) - \bar{u}_*^\pm(\ell(s), t)) \cdot (\bar{S}^+(\ell(s), s) + \bar{S}^-(\ell(s), s))$$

and this proves the Lemma. $\square$

Lemma 4

$$\left(\frac{d}{ds}\right)_t \int_0^{\ell(s)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{e}_2] dq = \int_0^{\ell(t)} [\dot{\bar{u}}_t \cdot \bar{I}_t \bar{e}_2] dq \quad (3.37)$$

$$\left(\frac{d}{ds}\right)_t \int_{\partial B} (\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{n} ds(x) = \int_{\partial B} \dot{\bar{u}}_t \cdot \bar{I}_t \bar{n} ds(x) \quad (3.38)$$

Proof of Lemma 4: From (II), (xvi) - (xix) and Lemma A.9 it follows that

$$\frac{d}{ds} \int_0^{\ell(s)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{e}_2] dq = \int_0^{\ell(s)} [\dot{\bar{u}}_s \cdot \bar{I}_s \bar{e}_2 + (\bar{u}_s - \bar{u}_t) \cdot \dot{\bar{I}}_s \bar{e}_2] dq +$$

$$([(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_s \bar{e}_2](\ell(s))) c(s)$$

and by evaluating this relation at $s = t$ we get (3.37). The proof of (3.38) is trivial (since $\bar{u}$ and $\bar{I}$ are smooth on the fix curve ∂B). $\square$

Lemma 5

$$\left(\frac{d}{ds}\right)_t (\bar{u}^\pm(\ell(s), s) - \bar{u}_*^\pm(\ell(s), t)) = \bar{c}(t) - (\bar{e}^\pm L^\pm)(\ell(t), t) c(t) \quad (3.39)$$

Proof of Lemma 5: We have

$$\frac{d}{ds} \bar{u}_*^\pm(\ell(s), t) = \frac{\partial \bar{u}_*^\pm}{\partial q}(\ell(s), t) c(s)$$

and by (3.25)

$$\frac{d}{ds} \bar{u}^{\pm}(\ell(s), s) = \bar{c}(s) - \bar{e}_1 c(s)$$

hence

$$\frac{d}{ds} (\bar{u}^{\pm}(\ell(s), s) - \bar{u}_*^{\pm}(\ell(s), t)) = \bar{c}(s) - (\bar{e}_1 + \frac{\partial \bar{u}_*^{\pm}}{\partial q}(\ell(s), t)) c(s)$$

and by evaluating this at $s = t$ we get (3.39). $\square$

Remark: Note that

$$\bar{e}^+(\ell(t), t) = \bar{e}^-(\ell(t), t), \quad L^+(\ell(t), t) = L^-(\ell(t), t) \quad (3.40)$$

Lemma 6

$$\left(\frac{d}{ds}\right)_t \int_B (\bar{u}_s - \bar{u}_t) \cdot \bar{b}_s \rho \, da(x) = \int_B \dot{\bar{u}}_t \cdot \bar{b}_t \rho \, da(x) \quad (3.41)$$

Proof of Lemma 6: From (xii) and (3.27) it follows that

$$\frac{d}{ds} ((\bar{u}_s - \bar{u}_t) \cdot \bar{b}_s \rho) = \dot{\bar{u}}_s \cdot \bar{b}_s \rho + (\bar{u}_s - \bar{u}_t) \cdot \dot{\bar{b}}_s \rho \in L^1(B) \text{ uniformly in time}$$

and from Lemma A.2 we have

$$\frac{d}{ds} \int_B (\bar{u}_s - \bar{u}_t) \cdot \bar{b}_s \rho \, da(x) = \int_B (\dot{\bar{u}}_s \cdot \bar{b}_s + (\bar{u}_s - \bar{u}_t) \cdot \dot{\bar{b}}_s) \rho \, da(x)$$

and (3.41) is proved by evaluating this relation at $s = t$. $\square$

Lemma 7

$$\int_B \dot{\bar{u}}_t \cdot \mathbb{I}_t \bar{n} \, ds(x) - \int_0^{\ell(t)} [\dot{\bar{u}}_t \cdot \mathbb{I}_t \bar{e}_2] \, dq + \int_B \dot{\bar{u}}_t \cdot \bar{b}_t \rho \, da(x) -$$

$$\dot{\bar{u}}^+(\ell(t), t) \cdot \bar{S}^+(\ell(t), t) - \dot{\bar{u}}^-(\ell(t), t) \cdot \bar{S}^-(\ell(t), t) =$$

$$\left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s (\nabla \dot{\bar{u}}_\lambda \cdot \mathbb{I}_s + \dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_s \rho) \, d\lambda \right) da(x) \quad (3.42)$$

Proof of Lemma 7: From (3.35), (3.37) - (3.39) and (xx) we have

$$\begin{aligned}
\left(\frac{d}{ds}\right)_t^x \int_B ((\nabla \bar{u}_s - \nabla \bar{u}_t) \cdot \bar{I}_s + (\bar{u}_s - \bar{u}_t) \cdot \ddot{\bar{u}}_s \rho) da(x) &= \int_B \dot{\bar{u}}_t \cdot \bar{b}_t \rho da(x) + \\
\int_{\partial B} \dot{\bar{u}}_t \cdot \bar{I}_t \bar{n} ds(x) - \int_0^{\ell(t)} [\dot{\bar{u}}_t \cdot \bar{I}_t \bar{e}_2] dq - \\
(\bar{c}(t) - (\bar{e}^\pm L^\pm)(\ell(t), t) c(t)) \cdot (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) & \quad (3.43)
\end{aligned}$$

But, from (3.24) and (3.40), we have

$$\begin{aligned}
(\bar{c}(t) - (\bar{e}^\pm L^\pm)(\ell(t), t) c(t)) \cdot (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) &= \\
(\bar{c}(t) \cdot \bar{e}^\pm(\ell(t), t) - L^\pm(\ell(t), t) c(t)) (S^+(\ell(t), t) + S^-(\ell(t), t)) &= \\
\dot{\bar{u}}^\pm(\ell(t), t) \cdot \bar{e}^\pm(\ell(t), t) (S^+(\ell(t), t) + S^-(\ell(t), t)) &= \\
\dot{\bar{u}}^\pm(\ell(t), t) \cdot (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) & \quad (3.44)
\end{aligned}$$

For points $x \in B \setminus C_T$ we have ($C^* = C_T$)

$$(\nabla \bar{u}_s - \nabla \bar{u}_t) \cdot \bar{I}_s = \int_t^s \nabla \dot{\bar{u}}_\lambda \cdot \bar{I}_s d\lambda, \quad (\bar{u}_s - \bar{u}_t) \cdot \ddot{\bar{u}}_s \rho = \int_t^s \dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_s \rho d\lambda$$

and since C_T is a set of area-measure zero we may conclude that

$$\begin{aligned}
\int_B ((\nabla \bar{u}_s - \nabla \bar{u}_t) \cdot \bar{I}_s + (\bar{u}_s - \bar{u}_t) \cdot \ddot{\bar{u}}_s \rho) da(x) &= \\
\int_B \left(\int_t^s (\nabla \dot{\bar{u}}_\lambda \cdot \bar{I}_s + \dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_s \rho) d\lambda \right) da(x) & \quad (3.45)
\end{aligned}$$

The Lemma now follows from (3.43) - (3.45). $\square$

Lemma 8

$$\begin{aligned}
\left(\frac{d}{ds}\right)_t \int_B \left(\frac{1}{2} \dot{\bar{u}}_s \cdot \dot{\bar{u}}_s + w_s \right) \rho da(x) &= \\
\left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s (\dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_\lambda \rho + \nabla \dot{\bar{u}}_\lambda \cdot \bar{I}_\lambda) d\lambda \right) da(x) & \quad (3.46)
\end{aligned}$$

Proof of Lemma 8: For points $x \in B \setminus C_T$ we have

$$\left(\frac{1}{2} \dot{\bar{u}}_s \cdot \dot{\bar{u}}_s + w_s\right) \rho - \left(\frac{1}{2} \dot{\bar{u}}_t \cdot \dot{\bar{u}}_t + w_t\right) \rho = \int_t^s (\dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_\lambda + \dot{w}_\lambda) \rho d\lambda \quad (3.47)$$

where, according to (2.28)₁,

$$\dot{w}_\lambda \rho = \bar{\Gamma}_\lambda \cdot \nabla \dot{\bar{u}}_\lambda$$

The Lemma now follows by integrating (3.47) over B and then differentiating with respect to s . $\square$

That (VI)' is equivalent to (VI) is now a simple consequence of Lemma 7 and Lemma 8 and this completes the proof of Theorem 1. $\square$

Remark 1: It is a simple matter to confirm that the method used in the proof of Theorem 1 also works if the crack is retreating, i.e. if $c(t) \leq 0$ for $t \in [0, T]$. The formula obtained in this case is similar to (VI)'. We only have to change the right-derivative (at $s = t$) for the left-derivative.

Remark 2: It is a consequence of the proof of Theorem 1 (Lemma 1 & 2) that if $\bar{u}$ is a displacement field satisfying: (i) - (ix), (xi) - (xxi) (item (x) is omitted), then the mappings

$$[0, T] \ni t \leadsto \int_B \dot{\bar{u}} \rho da(x), \quad [0, T] \ni t \leadsto \int_B (\chi - x_0) * \dot{\bar{u}} \rho da(x) \quad (3.48)$$

are C^1 and their derivatives are given by formulas (3.26) and (3.30), respectively. Thus assumption (x), in the definition of $\mathcal{D}'$, is partly redundant. By introducing some extra conditions on $\bar{u}$ the item (x) may be left out completely.

Denote by $\mathcal{D}'_e$ the set of displacement fields satisfying: (i) - (ix), (xi) - (xxi),

(xxii)_e $\bar{u}''$ and $\nabla \bar{u}' \in L^2(B)$ both uniformly in time

and

(xxiii)_e w and $\bar{e}_1 \cdot \bar{w} \in L^1(B)$, $\bar{\Gamma} \in L^2(B)$ all uniformly in time, where

$$\bar{w}(x, t) \equiv \frac{\partial w}{\partial x}(\nabla \bar{u}(x, t), x), \quad x \in B \setminus C_t \quad (3.49)$$

We then have

Proposition 1 If ρ is a C^1 -function then

$$\mathcal{D}'_e \subset \mathcal{D}' \quad (3.50)$$

Proof: We only have to prove that if $\bar{u} \in \mathcal{D}'_e$ then the mapping

$$[0, T] \ni t \mapsto \int_B \left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \rho \, da(x) \quad (3.51)$$

is C^1 . Thus, if $\bar{u} \in \mathcal{D}'_e$ then, from (xiii), (xiv) and (xxiii)_e, it follows that

$$\left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \rho = \left(\frac{1}{2} \bar{u}' \cdot \bar{u}' - \bar{u}' \cdot \partial_c \bar{u} + \frac{1}{2} \partial_c \bar{u} \cdot \partial_c \bar{u} + w \right) \rho \in L^1(B) \text{ uniformly in time}$$

Furthermore

$$\begin{aligned} \left(\left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \rho \right)' &= (\bar{u}' \cdot \bar{u}'' - \bar{u}'' \cdot \partial_c \bar{u} - \bar{u}' \cdot \partial_c \bar{u}' - \bar{u}' \cdot \partial_c \dot{\bar{u}} + \partial_c \bar{u} \cdot \partial_c \bar{u}' + \\ &\quad \partial_c \bar{u} \cdot \partial_c \bar{u}' + w') \rho + \left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \partial_c \rho \end{aligned} \quad (3.52)$$

We need

Lemma 9

$$w' = \frac{1}{\rho} \underline{I} \cdot \nabla \bar{u}' + c \bar{e}_1 \cdot \bar{w} \quad (3.53)$$

Proof of Lemma 9: From (2.28)₁ and (3.49) it follows that

$$\begin{aligned} w' &= \dot{w} + \partial_c w = \frac{1}{\rho} \underline{I} \cdot \nabla \dot{\bar{u}} + \frac{1}{\rho} \underline{I} \cdot \partial_c \nabla \bar{u} + c \bar{e}_1 \cdot \bar{w} = \frac{1}{\rho} \underline{I} \cdot (\nabla \dot{\bar{u}} + \partial_c \nabla \bar{u}) + \\ c \bar{e}_1 \cdot \bar{w} &= \frac{1}{\rho} \underline{I} \cdot \nabla \bar{u}' + c \bar{e}_1 \cdot \bar{w} \end{aligned}$$

□

From (3.52), (xiii), (xiv), (xxii)_e, (xxiii)_e and (3.53) it follows that

$$\left(\left(\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w \right) \rho \right)' \in L^1(B) \text{ uniformly in time}$$

An application of Lemma A.3 (with $\varphi = (\frac{1}{2}\dot{\bar{u}} \cdot \dot{\bar{u}} + w) \rho$) now proves the Proposition. $\square$

The derivative of function (3.51) is given by formula A.4

$$\begin{aligned} \frac{d}{dt} \int_B (\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w) \rho \, da(x) &= \int_B ((\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w) \rho)' \, da(x) - \\ & c \int_{\partial B} (\frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} + w) \rho \, \bar{e}_1 \cdot \bar{n} \, ds(x) \end{aligned} \quad (3.54)$$

but this expression will not be used in our further discussions.

Remark 3: Note that from the proof of Proposition 1 it follows that if $\bar{u} \in \mathcal{D}'_e$ then both the kinetic energy and the strain energy

$$[0, T] \ni t \leadsto \int_B \frac{1}{2} \dot{\bar{u}} \cdot \dot{\bar{u}} \rho \, da(x) \quad \text{and} \quad [0, T] \ni t \leadsto \int_B w \rho \, da(x) \quad (3.55)$$

are C^1 -functions.

Remark 4: From our previous discussions it is clear that B in (VI)' may be replaced by any region (subbody) P containing the crack-tip.

The Rice fracture criterion was first presented in Rice [1] under assumptions more special than those used in this paper. Rice [1] considers infinitesimal strains, time-independent surface-tractions and traction-free crack-surfaces ($\bar{S}^\pm = \bar{0}$).

Crucial in our presentation are the restrictions concerning the kinematical behaviour at the crack-tip. These assumptions enforce, as we have seen, tangential crack-surfaces at the crack-tip but they allow for velocities $\dot{\bar{u}}^\pm$ that may be singular at the crack-tip.

As an alternative to the set $\mathcal{D}'$ we denote by $\mathcal{D}'_a$ the set of displacement fields satisfying: (i) - (xvi) and, with $\bar{u}_*^\pm$ defined as in (3.23),

$$\begin{aligned} \text{(xvii)}_a \quad \bar{u}_*^\pm(q, \cdot) &\in C^1([0, T]) \quad \text{for } q \in [0, \ell^*] \quad \text{and} \quad \dot{\bar{u}}_*^\pm(\cdot, t) \in C^0([0, \ell^*]) \\ &\text{for } t \in [0, T]. \end{aligned}$$

Remark: From (xvii)_a it follows that

$$\dot{\bar{u}}^+(\ell(t), t) = \dot{\bar{u}}^-(\ell(t), t) \quad (3.56)$$

but the crack-surfaces need not be tangential at the crack-tip.

Theorem 1a If the density ρ and the body force $\bar{b}$ are C^1 -functions and $\bar{u} \in \mathcal{D}'_a$ then the crack-tip conditions (IV) - (VI) are equivalent to (IV)', (V)' and the alternative form of the Rice fracture criterion

$$(\sigma^+(\ell(t), t) + \sigma^-(\ell(t), t)) c(t) =$$

$$\left(\frac{d}{ds}\right)_t^1 \int_B \left(\int_t^s (\bar{I}_t - \bar{I}_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda + \rho (\ddot{\bar{u}}_t - \ddot{\bar{u}}_\lambda) \cdot \dot{\bar{u}}_\lambda \right) d\lambda \, da(x) \quad (VI)'_a$$

where $\left(\frac{d}{ds}\right)_t^1$ denotes the left-derivative with respect to s evaluated at $s = t$.

Remark: Notice the differences between formulas (VI)' and (VI)'_a!

Proof: The demonstration of formulas (IV)' and (V)' is identical to that presented in the proof of Theorem 1. For the proof of (VI)'_a we need

Lemma 3a For $s \leq t$ we have

$$\begin{aligned} & \int_{\partial B} (\bar{u}_s - \bar{u}_t) \cdot \bar{I}_t \bar{n} \, ds(x) - \int_0^{\ell(t)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_t \bar{e}_2] \, dq = \\ & \int_B ((\nabla \bar{u}_s - \nabla \bar{u}_t) \cdot \bar{I}_t + (\bar{u}_s - \bar{u}_t) \cdot (\ddot{\bar{u}}_t - \ddot{\bar{b}}_t) \rho) \, da(x) + \\ & (\bar{u}_*^\pm(\ell(t), s) - \bar{u}^\pm(\ell(t), t)) \cdot (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) \end{aligned} \quad (3.57)$$

Proof of Lemma 3a: With obvious modifications we may use the method in the proof of Lemma 3. $\square$

Lemma 4a

$$\left(\frac{d}{ds}\right)_t^1 \int_0^{\ell(t)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_t \bar{e}_2] \, dq = \int_0^{\ell(t)} [\dot{\bar{u}}_t \cdot \bar{I}_t \bar{e}_2] \, dq \quad (3.58)$$

$$\left(\frac{d}{ds}\right)_t^1 \int_{\partial B} (\bar{u}_s - \bar{u}_t) \cdot \bar{I}_t \bar{n} \, ds(x) = \int_{\partial B} \dot{\bar{u}}_t \cdot \bar{I}_t \bar{n} \, ds(x) \quad (3.59)$$

Proof of Lemma 3a: From (II), (xvi) and (xvii)_a it follows that

$$\frac{d}{ds} \int_0^{\ell(t)} [(\bar{u}_s - \bar{u}_t) \cdot \bar{I}_t \bar{e}_2] dq = \int_0^{\ell(t)} [\dot{\bar{u}}_{*s} \cdot \bar{I}_t \bar{e}_2] dq$$

and by evaluating this at $s = t$ we get (3.58). $\square$

Lemma 6 a

$$\left(\frac{d}{ds}\right)_t \int_B (\bar{u}_s - \bar{u}_t) \cdot \bar{b}_t \rho da(x) = \int_B \dot{\bar{u}}_t \cdot \bar{b}_t \rho da(x) \quad (3.60)$$

Proof of Lemma 6 a: See the proof of Lemma 6! $\square$

Lemma 7 a

$$\int_{\partial B} \dot{\bar{u}}_t \cdot \bar{I}_t \bar{n} ds(x) - \int_0^{\ell(t)} [\dot{\bar{u}}_t \cdot \bar{I}_t \bar{e}_2] dq + \int_B \dot{\bar{u}}_t \cdot \bar{b}_t \rho da(x) -$$

$$\dot{\bar{u}}^\pm(\ell(t), t) \cdot (\bar{S}^+(\ell(t), t) + \bar{S}^-(\ell(t), t)) =$$

$$\left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s (\nabla \dot{\bar{u}}_\lambda \cdot \bar{I}_t + \dot{\bar{u}}_\lambda \cdot \ddot{\bar{u}}_t \rho) d\lambda \right) da(x) \quad (3.61)$$

Proof of Lemma 7 a: Differentiate (3.57) with respect to s and use (3.58) - (3.60). Proceed as in the proof of Lemma 7. $\square$

The equivalence of (VI) and (VI)'_a is now a simple consequence of Lemma 7 a and Lemma 8 and this completes the proof of Theorem 1 a. $\square$

4. QUASI-STATIC CONDITIONS

Throughout this section we assume that the density ρ and the body force $\bar{b}$ are C^1 -functions.

A displacement field $\bar{u} \in \mathcal{D}'$ is said to be quasi-static if

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = 0, \quad t \in [0, T] \quad (4.1)$$

Remark: It is a consequence of Theorem 1 that if $\bar{u} \in \mathcal{D}'$ is a solution to the boundary-initial-value problem then the right-derivative

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s ((\bar{I}_s - \bar{I}_\lambda) \cdot \nabla \dot{u}_\lambda + \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda) d\lambda \right) da(x) \quad (4.2)$$

exists (and is equal to $(\sigma^+(\ell(t), t) + \sigma^-(\ell(t), t))c(t)$). The existence of the separate derivatives

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s ((\bar{I}_s - \bar{I}_\lambda) \cdot \nabla \dot{u}_\lambda d\lambda) \right) da(x) \quad (4.3)$$

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) \quad (4.4)$$

is however not guaranteed.

Denote by $\mathcal{D}''$ the set of displacement fields $\bar{u} \in \mathcal{D}'$ satisfying the items (xxii) - (xxvii) stated below:

(xxii) The derivative (4.4) exists for $t \in [0, T]$.

(xxiii) $\ell \in C^3([0, T])$

(xxiv) The mapping in (iv) is C^3 .

(xxv) $\frac{\partial^2 \bar{u}}{\partial x_1^2} \in L^1(B)$ uniformly in time.

(xxvi) $\bar{u}'$ is bounded in the sense that $\sup_{\substack{x \in B \setminus C_t \\ t \in [0, T]}} |\bar{u}'(x, t)| < \infty$

(xxvii) $\bar{u}'''$, $\frac{\partial \bar{u}''}{\partial x_1}$ and $\frac{\partial^2 \bar{u}'}{\partial x_1^2} \in L^1(B)$ all uniformly in time.

We have

Theorem 2 If $\bar{u} \in \mathcal{D}''$ then

$$\begin{aligned} & \left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = \\ & - \left(\frac{d}{ds}\right)_t \int_B \frac{\rho}{2} |\partial_{c(s)}(\bar{u}_s - \bar{u}_t)|^2 da(x) \leq 0, \quad t \in [0, T[\end{aligned} \quad (4.5)$$

where $\partial_{c(s)} \equiv c(s) \frac{\partial}{\partial x_1}$ (cf. (3.4)).

Proof: We have, for $x \in B \setminus C_T$,

$$\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda = \frac{\rho}{2} (Q + (\bar{u}_s - \bar{u}_t) \cdot \partial_{c(s)}^2 (\bar{u}_s - \bar{u}_t)) \quad (4.6)$$

where $\partial_{c(s)}^2 \equiv c(s)^2 \frac{\partial^2}{\partial x_1^2}$ (cf. (3.5)) and

$$\begin{aligned} Q &= Q(x, s, t) \equiv 2 \ddot{u}_s(x) \cdot (\bar{u}_s(x) - \bar{u}_t(x)) - (\dot{u}_s^2(x) - \dot{u}_t^2(x)) - \\ & (\bar{u}_s(x) - \bar{u}_t(x)) \cdot \partial_{c(s)}^2 (\bar{u}_s(x) - \bar{u}_t(x)) \end{aligned} \quad (4.7)$$

We have

Lemma 10

$$\left(\frac{d}{ds}\right)_t \int_B \frac{\rho}{2} Q da(x) = 0 \quad (4.8)$$

Proof of Lemma 10: A simple calculation, using formula (3.8)₂, shows that if $\bar{u} \in \mathcal{D}''$ then $(\partial_c = \partial_{c(s)}, \partial_c^2 = \partial_{c(s)}^2, \text{etc.})$

$$\frac{\partial Q}{\partial s} = R + \partial_c S \quad (4.9)$$

where

$$R = R(x, s, t) \equiv 2 (\bar{u}_s'''(x) - 3 \partial_c \bar{u}_s''(x) + 3 \partial_c^2 \bar{u}_s'(x) - 3 \partial_c \bar{u}_s'(x) +$$

$$3 \partial_c \partial_c \bar{u}_s(x) - \partial_c \bar{u}_s(x) \cdot (\bar{u}_s(x) - \bar{u}_t(x)) - \bar{u}_s'(x) \cdot \partial_c (\bar{u}_s(x) - \bar{u}_t(x)) - \\ (\bar{u}_s'(x) - \bar{u}_t'(x)) \cdot \partial_c \bar{u}'(x) - 2 (\bar{u}_s(x) - \bar{u}_t(x)) \cdot \partial_c \partial_c (\bar{u}_s(x) - \bar{u}_t(x)) \quad (4.10)$$

and

$$S = S(x, s, t) \equiv - (\bar{u}_s(x) - \bar{u}_t(x)) \cdot \partial_c^2 \bar{u}_s(x) + \partial_c \bar{u}_s(x) \cdot \partial_c (\bar{u}_s(x) - \bar{u}_t(x))$$

$$\text{Note that} \quad (4.11)$$

$$R(x, t, t) = 0 \quad \text{and} \quad S(x, t, t) = 0 \quad x \in B \setminus C_t \quad (4.12)$$

From (4.9) it follows that

$$\frac{\partial}{\partial s} \left(\frac{\rho}{2} Q \right) = \frac{1}{2} (\rho R - S \partial_c \rho) + \partial_c \left(\frac{\rho}{2} S \right) \quad (4.13)$$

where, since $\bar{u} \in \mathcal{D}''$,

$$\frac{\rho}{2} Q, \quad \frac{1}{2} (\rho R - S \partial_c \rho) \in L^1(B) \quad \text{and} \quad \frac{\rho}{2} S \in L^1(\partial B) \quad \text{all uniformly in time (in} \\ s \in [0, T]). \quad (4.14)$$

An application of Lemma A.4 gives (t is fixed)

$$\frac{d}{ds} \int_B \frac{\rho}{2} Q \, da(x) = \int_B \frac{1}{2} (\rho R - S \partial_c \rho) \, da(x) + c \int_{\partial B} \frac{\rho}{2} S \bar{e}_1 \cdot \bar{n} \, ds(x) \quad (4.15)$$

and by evaluating this at $s = t$ the Lemma follows from (4.12). $\square$

From (4.6) and Lemma 10 we get

$$\left(\frac{d}{ds} \right)_t^r \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda \, d\lambda \right) da(x) = \\ \left(\frac{d}{ds} \right)_t^r \int_B \frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c^2 (\bar{u}_s - \bar{u}_t) \, da(x) \quad (4.16)$$

But

$$\frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c^2 (\bar{u}_s - \bar{u}_t) = \partial_c \left(\frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \right) -$$

$$\frac{1}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho - \frac{\rho}{2} \partial_c (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \quad (4.17)$$

and we have ($\partial_c = \partial_c(s)$, $\partial_c^2 = \partial_c^2(s)$, etc.)

Lemma 11

$$\left(\frac{d}{ds}\right)_t \int_B \partial_c \left(\frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t)\right) da(x) = 0 \quad (4.18)$$

Proof of Lemma 11: Since $\bar{u} \in \mathcal{D}''$ we have (from (4.17))

$$\partial_c \left(\frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t)\right) \in L^1(B) \quad (\text{uniformly in time})$$

and by using Lemma A.5 we find that (t is fixed)

$$\begin{aligned} \frac{d}{ds} \int_B \partial_c \left(\frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t)\right) da(x) = \\ \dot{c} \int_B \frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \bar{e}_1 \cdot \bar{n} ds(x) + c \int_B \frac{\rho}{2} (\dot{\bar{u}}_s \cdot \partial_c (\bar{u}_s - \bar{u}_t) + \\ (\bar{u}_s - \bar{u}_t) \cdot (\partial_c (\dot{\bar{u}}_s - \bar{u}_t) + \partial_c \dot{\bar{u}}_s)) \bar{e}_1 \cdot \bar{n} ds(x) \end{aligned}$$

and the Lemma follows by evaluating this relation at $s = t$. $\square$

Lemma 12

$$\left(\frac{d}{ds}\right)_t \int_B \frac{1}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho da(x) = 0 \quad (4.19)$$

Proof of Lemma 12: Since $\bar{u} \in \mathcal{D}''$ we have

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{1}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho\right) = \frac{1}{2} (\dot{\bar{u}}_s \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho + \\ (\bar{u}_s - \bar{u}_t) \cdot (\partial_c (\bar{u}_s - \bar{u}_t) + \partial_c \dot{\bar{u}}_s) \partial_c \rho + (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho) \in L^1(B) \end{aligned}$$

uniformly in time.

($\partial_c \dot{\bar{u}}_s = \partial_c \bar{u}' - \partial_c^2 \bar{u}_s$) and thus, from Lemma A.2, we find that (t fixed)

$$\frac{d}{ds} \int_B \frac{1}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho da(x) =$$

$$\int_B \frac{\partial}{\partial s} \left(\frac{1}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) \partial_c \rho \right) da(x)$$

and the Lemma follows by evaluating this at $s = t$. $\square$

Hence, from (4.16) - (4.19), we get

$$\begin{aligned} & \left(\frac{d}{ds} \right)_t^r \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = \\ & - \left(\frac{d}{ds} \right)_t^r \int_B \frac{\rho}{2} \partial_c (\bar{u}_s - \bar{u}_t) \cdot \partial_c (\bar{u}_s - \bar{u}_t) da(x) \end{aligned}$$

and this proves the equality in (4.5). The inequality is established by considering, for fixed $t \in [0, T]$, the function

$$[0, T] \ni s \mapsto g(s) = - \int_B \frac{\rho}{2} |\partial_c(s) (\bar{u}_s - \bar{u}_t)|^2 da(x) \quad (4.20)$$

It is obvious that

$$g(s) \leq 0 \text{ for } s \in [0, T] \text{ and } g(t) = 0$$

and since g has a right-derivative at $s = t$ we may conclude that

$$\left(\frac{d}{ds} \right)_t^r g \leq 0, \quad t \in [0, T[$$

This completes the proof of Theorem 2. $\square$

Corollary 1 If $\bar{u} \in \mathcal{D}''$ and the function g , defined in (4.20), is differentiable at $s = t$, then $\bar{u}$ is quasi-static.

Proof: Obvious. $\square$

Remark: Quasi-static conditions may be established in several ways. For instance, if $\bar{u} \in \mathcal{D}''$ and

$$\frac{\partial^3 \bar{u}}{\partial x_1^3} \in L^1(B) \text{ uniformly in time} \quad (4.21)$$

then, since

$$\ddot{\bar{u}} \in L^1(B) \text{ uniformly in time}$$

and (t fixed)

$$\frac{d}{ds} \int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda = \rho \ddot{u}_s \cdot (\bar{u}_s - \bar{u}_t) \in L^1(B) \text{ uniformly in time}$$

we have, from Lemma A.2,

$$\frac{d}{ds} \int_B \left(\int_t^s \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = \int_B \rho \ddot{u}_s \cdot (\bar{u}_s - \bar{u}_t) da(x)$$

and it follows that $\bar{u}$ is quasi-static. Another possibility is to assume that $\bar{u} \in \mathcal{D}''$ satisfies

$$\frac{\partial^2 \bar{u}}{\partial x_1^2}, \bar{u}'' \text{ and } \frac{\partial \bar{u}'}{\partial x_1} \in L^2(B) \text{ all uniformly in time} \quad (4.22)$$

and apply Lemma A.3 (for fixed t) to the field $\varphi_s = \rho (\ddot{u}_s \cdot (\bar{u}_s - \bar{u}_t) - \frac{1}{2}(\dot{u}_s - \dot{u}_t))$. Note that if $\bar{u} \in \mathcal{D}''$ satisfies (4.22) then

$$\ddot{u} \in L^2(B) \text{ uniformly in time}$$

In connection with Theorem 1a and the alternative form of the Rice fracture criterion we say that a displacement field $\bar{u} \in \mathcal{D}'_a$ is quasi-static if

$$\left(\frac{d}{ds} \right)_t \int_B \left(\int_t^s \rho (\ddot{u}_t - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = 0, \quad t \in [0, T] \quad (4.23)$$

Our next task will be to prove, for this case, a result analogous to the one stated in Theorem 2. For this we need the following definition:

For a family of fields $\varphi(\cdot, t) : B \subset C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ we write

$$\varphi \in L^2(B) \text{ uniformly in time } \underline{\text{in the weak sense}} \quad (4.24)$$

if $\varphi \cdot f \in L^1(B)$ uniformly in time for every (time-independent) mapping $f : B \rightarrow \mathbb{R}, \mathbb{R}^2$ such that $f \in L^2(B)$.

Denote by $\mathcal{D}''_a$ the set of displacement fields $\bar{u} \in \mathcal{D}'_a$ satisfying the items $(xviii)_a - (xxv)_a$ stated below:

(xviii)_a The derivative

$$\left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s \rho (\ddot{u}_t - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x)$$

exists for $t \in]0, T]$

(xix)_a $\equiv$ (xxiii)

(xx)_a $\equiv$ (xxiv)

(xxi)_a $\bar{u}$ is continuous at $t \in [0, T]$ in the sense that

$$\lim_{s \rightarrow t} \sup_{x \in B \setminus C_T} |\bar{u}(x, s) - \bar{u}(x, t)| = 0$$

(xxii)_a $\equiv$ (xxv)

(xxiii)_a $\equiv$ (xxvi)

(xxiv)_a $\bar{u}''$ and $\frac{\partial \bar{u}'}{\partial x_1} \in L^2(B)$ both uniformly in time in the weak sense.

(xxv)_a $\equiv$ (xxvii)

We have

Theorem 2 a If $\bar{u} \in \mathcal{D}_a''$ then

$$\begin{aligned} \left(\frac{d}{ds}\right)_t \int_B \left(\int_t^s \rho (\ddot{u}_t - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda \right) da(x) = \\ \left(\frac{d}{ds}\right)_t \int_B \frac{\rho}{2} |\partial_{C(s)}(\bar{u}_s - \bar{u}_t)|^2 da(x) \leq 0, \quad t \in]0, T] \end{aligned} \quad (4.25)$$

Proof: We have, for $x \in B \setminus C_T$,

$$\begin{aligned} \int_t^s \rho (\ddot{u}_t - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda = \frac{\rho}{2} (Q - T - 2(\bar{u}_s - \bar{u}_t) \cdot (\partial_{C(s)}^2 \bar{u}_t - \partial_{C(t)}^2 \bar{u}_t) - \\ (\bar{u}_s - \bar{u}_t) \cdot \partial_{C(s)}^2 (\bar{u}_s - \bar{u}_t)) \end{aligned} \quad (4.26)$$

where Q is defined in (4.7) and

$$T = T(x, s, t) \equiv 2 (\bar{u}_s''(x) - 2 \partial_{c(s)} \bar{u}_s'(x) - \partial_{\dot{c}(s)} \bar{u}_s(x) - (\bar{u}_t''(x) - 2 \partial_{c(t)} \bar{u}_t'(x) - \partial_{\dot{c}(t)} \bar{u}_t(x))) \cdot (\bar{u}_s(x) - \bar{u}_t(x)) \quad (4.27)$$

and

Lemma 13

$$\left(\frac{d}{ds}\right)_t \int_B \frac{\rho}{2} T da(x) = 0 \quad (4.28)$$

Proof of Lemma 13: Since $\bar{u} \in \mathcal{D}_a''$ we have (where the convected derivative ' is taken with respect to s keeping t fixed)

$$\begin{aligned} \frac{T'}{2} &= (\bar{u}_s''' - 2 \partial_{c(s)} \bar{u}_s'' - 3 \partial_{\dot{c}(s)} \bar{u}_s' - \partial_{\ddot{c}(s)} \bar{u}_s - (\partial_{c(s)} \bar{u}_t'' - 2 \partial_{c(s)} \partial_{c(t)} \bar{u}_t' - \\ &\partial_{c(s)} \partial_{\dot{c}(t)} \bar{u}_t)) \cdot (\bar{u}_s - \bar{u}_t) + (\bar{u}_s'' - 2 \partial_{c(s)} \bar{u}_s' - \partial_{\dot{c}(s)} \bar{u}_s - (\bar{u}_t'' - 2 \partial_{c(t)} \bar{u}_t' - \\ &\partial_{\dot{c}(t)} \bar{u}_t)) \cdot (\bar{u}_s' - \partial_{c(s)} \bar{u}_t) \in L^1(B) \text{ uniformly in time} \end{aligned}$$

(Note that from (xxiv)_a it follows that $\bar{u}_s'' \cdot \partial_{c(s)} \bar{u}_t$ and $\partial_{c(s)} \bar{u}' \cdot \partial_{c(s)} \bar{u}_t \in L^1(B)$ uniformly in time). Thus, from Lemma A.3, we have

$$\frac{d}{ds} \int_B \frac{\rho}{2} T da(x) = \int_B \frac{1}{2} (\rho' T + \rho T') da(x) - c \int_B \frac{\rho}{2} T \bar{e}_1 \cdot \bar{n} ds(x) \quad (4.29)$$

The Lemma now follows from (4.29) and the fact that

$$T(x, t, t) = 0, \quad T'(x, t, t) = 0, \quad x \in B \setminus C_T \quad (4.30)$$

□

Lemma 14

$$\left(\frac{d}{ds}\right)_t \int_B (\bar{u}_s - \bar{u}_t) \cdot (\partial_{c(s)}^2 \bar{u}_t - \partial_{c(t)}^2 \bar{u}_t) da(x) = 0 \quad (4.31)$$

Proof of Lemma 14:(cf. Gurtin & Yatomi [5], Theorem 3). Consider, for fixed t , the function

$$[0, T] \ni s \mapsto h(s) = \int_B \rho (\bar{u}_s - \bar{u}_t) \cdot (\partial_{c(s)}^2 \bar{u}_t - \partial_{c(t)}^2 \bar{u}_t) da(x)$$

Since $h(t) = 0$ we may write

$$\left(\frac{d}{ds}\right)_t h = \lim_{s \rightarrow t} \frac{h(s)}{s - t}$$

But we have the estimate

$$\left|\frac{h(s)}{s - t}\right| \leq \left(\sup_{x \in B \setminus C_T} |\bar{u}_s(x) - \bar{u}_t(x)|\right) \left|\frac{c(s)^2 - c(t)^2}{s - t}\right| \int_B \left|\frac{\partial^2 \bar{u}}{\partial x_1^2}\right| da(x)$$

and from (xxi)_a and

$$\lim_{s \rightarrow t} \left|\frac{c(s)^2 - c(t)^2}{s - t}\right| = 2 |c(t) \dot{c}(t)|$$

it follows that

$$\lim_{s \rightarrow t} \frac{h(s)}{s - t} = 0$$

and this proves the Lemma. $\square$

Hence, from (4.26), Lemma 10, Lemma 13 and Lemma 14, we get

$$\begin{aligned} \left(\frac{d}{ds}\right)_t \frac{1}{B} \int_t^s \left(\int \rho (\ddot{u}_s - \ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda\right) da(x) = \\ - \left(\frac{d}{ds}\right)_t \frac{1}{B} \int \frac{\rho}{2} (\bar{u}_s - \bar{u}_t) \cdot \partial_{c(s)}^2 (\bar{u}_s - \bar{u}_t) da(x) \end{aligned}$$

Following the proof of Theorem 2 (from (4.16) and onwards) we may now demonstrate (4.25). $\square$

Corollary 1a If $\bar{u} \in \mathcal{D}_a''$ and the function g , defined in (4.20), is differentiable at $s = t$, then $\bar{u}$ is quasi-static.

Remark: Note that we may use Lemma A.3 to prove that if $\bar{u} \in \mathcal{D}_a''$ satisfies

$$\frac{\partial^2 \bar{u}}{\partial x_1^2} \in L^2(B), \quad \frac{\partial^3 \bar{u}}{\partial x_1^3} \in L^1(B) \quad \text{and} \quad \bar{u}'', \quad \frac{\partial \bar{u}'}{\partial x_1} \in L^2(B) \quad \text{both uniformly in time}$$

(4.32)

then $\bar{u}$ is quasi-static.

5. STABILITY IN THE SENSE OF DRUCKER AND CRACK PROPAGATION.

The hyperelastic bulk material under consideration is said to be stable in the sense of Drucker if for every deformation $\bar{u} \in \mathcal{D}$

$$\int_a^b (I_\lambda - I_a) \cdot \nabla \dot{\bar{u}}_\lambda d\lambda \geq 0 \quad (5.1)$$

for every $a, b \in [0, T]$ and $x \in B \setminus C^*$.

Remark 1: See Rice [1] and Drucker [4] for a discussion on this stability concept.

Remark 2: Note that if the bulk elastic potential W is assumed to be convex, i.e.

$$W(H_2, x) - W(H_1, x) \geq \frac{\partial W}{\partial H}(H_1, x) \cdot (H_2 - H_1), \quad H_1, H_2 \in \text{HEnd}(\mathbb{R}^2) \quad (5.2)$$

$x \in B$, then the bulk material is stable in the sense of Drucker. The assumption that W is convex is however in conflict with the requirement that W be objective (cf. (2.17) and see Truesdell & Noll [7], p. 163).

We have

Proposition 2 Let B be a hyperelastic body with bulk material stable in the sense of Drucker. Then if

a) $\bar{u} \in \mathcal{D}''$ we have

$$\left(\frac{d}{ds}\right)_t^r \int_B \left(\int_t^s (I_s - I_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda d\lambda \right) da(x) \geq 0, \quad t \in [0, T] \quad (5.3)$$

b) $\bar{u} \in \mathcal{D}_a''$ we have

$$\left(\frac{d}{ds}\right)_t^1 \int_B \left(\int_t^s (I_t - I_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda d\lambda \right) da(x) \geq 0, \quad t \in [0, T] \quad (5.4)$$

Proof: a) Consider, for fixed $t \in [0, T]$, the function

$$[0, T] \ni s \mapsto i(s) = \int_B \left(\int_t^s (I_s - I_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda d\lambda \right) da(x) \quad (5.5)$$

It is obvious that $i(t) = 0$ and from (5.1) we may conclude that

$$i(s) \geq 0 \quad \text{for } s \in [0, T]$$

Then, since i has a right-derivative at $s = t$, we obtain (5.3). $\square$

b) Analogous. $\square$

Remark: Note that if the function i , defined in (5.5), is differentiable at $s = t$ then the derivatives in (5.3) and (5.4) are equal to zero.

The elastic surface material, represented by the potential Σ is said to have positive surface strain-energy if

$$\Sigma(L, q) > 0, \quad L > 0 \quad (5.6)$$

$$q \in [0, \ell^*].$$

Put

$$] \sigma[(t) \equiv \sigma^+(\ell(t), t) + \sigma^-(\ell(t), t)$$

$$k(t) \equiv \left(\frac{d}{ds} \right)_t^r \int_B \frac{\rho}{2} \left| \frac{\partial}{\partial x_1} (\bar{u}_s - \bar{u}_t) \right|^2 da(x) \quad (5.7)$$

$$w(t) \equiv \left(\frac{d}{ds} \right)_t^r \int_B \left(\int_t^s (I_s - I_\lambda) \cdot \nabla \dot{\bar{u}}_\lambda d\lambda \right) da(x)$$

Theorem 3 Let B be a hyperelastic body with bulk material stable in the sense of Drucker and with positive surface strain-energy. Let $\bar{u} \in \mathcal{D}''$ be a solution to the boundary-initial value problem. Then if

a) $k(t) > 0$ we have

$$c(t) = \frac{] \sigma[(t)}{2k(t)} \left(\sqrt{1 + \frac{4k(t)w(t)}{([\sigma[(t))^2}} - 1 \right) \quad (5.8)$$

b) $k(t) = 0$ we have

$$c(t) = \frac{w(t)}{] \sigma[(t)} \quad (5.9)$$

Proof: If $\bar{u} \in \mathcal{D}''$ then the Rice fracture criterion may be written (see Theorem 1 and Theorem 2)

$$k(t) c(t)^2 + \int \sigma(t) c(t) = w(t) \quad (5.10)$$

where (see Theorem 2 and Proposition 2)

$$k(t) \geq 0, \quad \int \sigma(t) > 0 \quad \text{and} \quad w(t) \geq 0 \quad (5.11)$$

and the Theorem follows by solving the quadratic equation (5.10). $\square$

Remark: Note that if the hyperelastic bulk material is non-stable in the sense of Drucker, i.e. if for every deformation $\bar{u} \in \mathcal{D}$

$$\int_a^b (\mathbb{I}_\lambda - \mathbb{I}_a) \cdot \nabla \dot{\bar{u}}_\lambda \, d\lambda < 0 \quad (5.12)$$

for every $a, b \in [0, T]$, $x \in B \setminus C^*$ and if we have positive surface strain-energy then if $\bar{u}$ is a solution to the boundary-initial value problem within the class $\mathcal{D}''$ it corresponds to a resting crack, i.e. $\ell(t) = \ell_0$ for $t \in [0, T]$.

In the linearized theory the elastic material under consideration is defined by the elasticity field:

$$\Phi: B \rightarrow L(\text{End}(\mathbb{R}^2), \text{End}(\mathbb{R}^2)), \quad \Phi(x) \equiv \rho(x) \frac{\partial^2 W}{\partial H^2}(\underline{Q}, x) \quad (5.13)$$

the elasticity fields on the crack-surfaces

$$k: [0, \ell^*] \rightarrow \mathbb{R}, \quad k(q) \equiv \frac{\partial^2 \Sigma}{\partial L^2}(1, q) \quad (5.14)$$

the residual stresses

$$\mathbb{I}^{\text{res}}(x) \equiv \rho(x) \frac{\partial W}{\partial H}(\underline{Q}, x), \quad S^{\text{res}}(q) \equiv \frac{\partial \Sigma}{\partial L}(1, q) \quad (5.15)$$

and the residual strain-energy densities

$$w^{\text{res}}(x) \equiv W(\underline{Q}, x), \quad \sigma^{\text{res}}(q) \equiv \Sigma(1, q) \quad (5.16)$$

Note that

$$\underline{A} \cdot \Phi(x) [\underline{B}] = \underline{B} \cdot \Phi(x) [\underline{A}], \quad \underline{A}, \underline{B} \in \text{End}(\mathbb{R}^2) \quad (5.17)$$

We have (in the linearized theory)

$$\begin{aligned}
 \underline{I}(x, t) &= \Phi(x) [\nabla \bar{u}(x, t)] + \underline{I}^{\text{res}}(x) \\
 \rho(x) w(x, t) &= \frac{1}{2} \nabla \bar{u}(x, t) \cdot \Phi(x) [\nabla \bar{u}(x, t)] + \underline{I}^{\text{res}} \cdot \nabla \bar{u}(x, t) + w^{\text{res}}(x) \\
 \bar{S}^{\pm}(q, t) &= \bar{e}_1(k(q) \frac{\partial u_1^{\pm}}{\partial q}(q, t) + S^{\text{res}}(q)) \\
 \sigma^{\pm}(q, t) &= \frac{1}{2} k(q) \left(\frac{\partial u_1^{\pm}}{\partial q}(q, t) \right)^2 + S^{\text{res}}(q) \frac{\partial u_1^{\pm}}{\partial q} + \sigma^{\text{res}}(q)
 \end{aligned} \tag{5.18}$$

where $u_1^{\pm}(q, t) = \bar{e}_1 \cdot \bar{u}^{\pm}(q, t)$, $q \in [0, \ell(t)]$.

Remark: Note that in the linearized theory

$$\Sigma(L, q) = \frac{1}{2} k(q) L^2 + (S^{\text{res}}(q) - k(q)) L + \frac{1}{2} k(q) - S^{\text{res}}(q) + \sigma^{\text{res}}(q) \tag{5.19}$$

where L is the length-ratio in the linearized theory $L = 1 + \frac{\partial u_1^{\pm}}{\partial q}$.

We say that Φ is positive semi-definite if

$$\underline{A} \cdot \Phi(x) [\underline{A}] \geq 0, \quad \underline{A} \in \text{End}(\mathbb{R}^2) \tag{5.20}$$

We have

Proposition 3 a) If Φ is positive semi-definite then the hyperelastic bulk material in the linearized theory is stable in the sense of Drucker.

b) If

$$k(q) > 0 \quad \text{and} \quad (S^{\text{res}}(q))^2 - 2 k(q) \sigma^{\text{res}}(q) < 0 \quad q \in [0, \ell^*] \tag{5.21}$$

then the elastic surface material has positive surface strain-energy.

Proof: a) We have, for $\bar{u} \in \mathcal{D}$, $x \in B \setminus C^*$ and $a, b \in [0, T]$,

$$\begin{aligned}
 \int_a^b (\underline{I}_{\lambda} - \underline{I}_a) \cdot \nabla \bar{u}_{\lambda} \, d\lambda &= \rho (w_b - w_a) - \underline{I}_a \cdot (\nabla \bar{u}_b - \nabla \bar{u}_a) = \frac{1}{2} \nabla \bar{u}_b \cdot \Phi[\nabla \bar{u}_b] - \\
 &\frac{1}{2} \nabla \bar{u}_a \cdot \Phi[\nabla \bar{u}_a] - (\nabla \bar{u}_b - \nabla \bar{u}_a) \cdot \Phi[\nabla \bar{u}_a] = \frac{1}{2} (\nabla \bar{u}_a - \nabla \bar{u}_b) \cdot \Phi[\nabla \bar{u}_a - \nabla \bar{u}_b] \geq 0
 \end{aligned}$$

where we have used (5.17) and (5.20).

b) Since $k(q) > 0$ we have

$$\Sigma(L, q) = \frac{1}{2} k(q) \left(L + \frac{S^{\text{res}}(q) - k(q)}{k(q)} \right)^2 + \sigma^{\text{res}}(q) - \frac{(S^{\text{res}}(q))^2}{2 k(q)} > 0$$

where we have used (5.21). $\square$

APPENDIX

In this Appendix we present some auxiliary results used in the main text. Lemmas A.2 - A.4 are taken from Gurtin & Yatomi [5] (see also Lidström [6] for a discussion in the 3-dimensional context).

Throughout this Appendix we assume that $\ell \in C^1([0, T])$.

Lemma A.1 Let $\varphi_1(\cdot, t), \varphi_2(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ be two families of fields.

- a) If $\varphi_1 \in L^2(B)$ uniformly in time then $\varphi_1 \in L^1(B)$ uniformly in time.
- b) If $\varphi_1, \varphi_2 \in L^2(B)$ uniformly in time then $\varphi_1 \cdot \varphi_2 \in L^1(B)$ uniformly in time.
- c) If $\varphi_1 \in L^1(B)$ uniformly in time and

$$\sup_{\substack{x \in B \setminus C_t \\ t \in [0, T]}} |\varphi_2(x, t)| < \infty \quad (A.1)$$

then $\varphi_1 \cdot \varphi_2 \in L^1(B)$ uniformly in time.

Proof: Let B_δ , $\delta > 0$ be a one-parameter family of measurable subsets of B with $\text{area}(B \setminus B_\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

a) If $\varphi_1 \in L^2(B)$ uniformly in time then, from Hölder's inequality¹, we have

$$0 \leq \lim_{\delta \rightarrow 0} \sup_{t \in [0, T]} \int_{B \setminus B_\delta} |\varphi_1(x, t)| \, da(x) \leq$$

$$\left(\lim_{\delta \rightarrow 0} \sup_{t \in [0, T]} \int_{B \setminus B_\delta} |\varphi_1(x, t)|^2 \, da(x) \right)^{\frac{1}{2}} \left(\lim_{\delta \rightarrow 0} \text{area}(B \setminus B_\delta) \right)^{\frac{1}{2}} = 0$$

The proofs of b) and c) are equally simple. $\square$

Lemma A.2 Let $\varphi(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ be a family of fields satisfying condition (β) in section 3 and in addition

$$(\gamma) \quad \varphi \in L^1(B) \quad \text{and} \quad (\delta) \quad \dot{\varphi} \in L^1(B) \quad \text{uniformly in time}$$

then the function

$$[0, T] \ni t \mapsto \int_B \varphi(x, t) \, da(x) \quad (A.2)$$

¹ See Schwartz [10]

is C^1 and

$$\frac{d}{dt} \int_B \dot{\varphi} da(x) = \int_B \varphi da(x) \quad (A.3)$$

Proof: Let R_δ , $\delta > 0$ be a family of rectangles in $\mathbb{R}^2$ with two boundary-lines parallel and two perpendicular to the crack and with

$$\{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \ell^*, x_2 = 0\} \equiv C^* \subset \overset{\circ}{R}_\delta$$

for every $\delta > 0$. See Figure below! (ℓ^* is defined in (2.2))

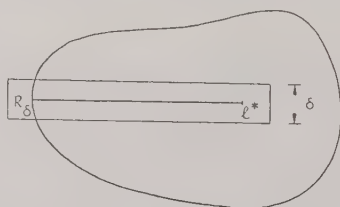


Figure A.1

Put $B_\delta \equiv B \setminus R_\delta$. Then it follows from (β) that the function $B_\delta \times [0, T] \ni (x, t) \mapsto \varphi(x, t)$ is C^1 and thus

$$\frac{d}{dt} \int_{B_\delta} \varphi da(x) = \int_{B_\delta} \dot{\varphi} da(x)$$

But, since $\text{area}(B \setminus B_\delta) \rightarrow 0$ as $\delta \rightarrow 0$, it follows from (δ) that

$$\frac{d}{dt} \int_{B_\delta} \varphi da(x) \rightarrow \int_B \dot{\varphi} da(x)$$

uniformly in $[0, T]$ as $\delta \rightarrow 0$. From (γ) we have

$$\int_{B_\delta} \varphi da(x) \rightarrow \int_B \varphi da(x)$$

as $\delta \rightarrow 0$ and the Lemma follows from standard theorems on uniform convergence (see Schwartz [10], chap. I, Theorems 39 and 43). $\square$

Remark: Lemma A.2 may be proved along the same lines even if the crack C is curved.

Lemma A.3 Let $\varphi(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ be a family of fields satisfying the conditions (α) , (β) in section 3 and (γ) in Lemma A.2 and in addition

(ϵ) $\varphi' \in L^1(B)$ and $\varphi|_{\partial B} \in L^1(\partial B)$, both uniformly in time.

Then the function (A.2) is C^1 and we have the formula

$$\frac{d}{dt} \int_B \varphi da(x) = \int_B \varphi' da(x) - c \int_{\partial B} \varphi \bar{e}_1 \cdot \bar{n} ds(x) \quad (A.4)$$

Proof: With R_δ and B_δ as in the proof of Lemma A.2 we have

$$\frac{d}{dt} \int_{B_\delta} \varphi da(x) = \int_{B_\delta} \dot{\varphi} da(x) = \int_{B_\delta} \varphi' da(x) - \int_{B_\delta} \partial_c \varphi da(x)$$

where we have used (3.3). By the divergence theorem we get

$$\int_{B_\delta} \partial_c \varphi da(x) = c \int_{\partial B_\delta} \varphi \bar{e}_1 \cdot \bar{n} ds(x) = c \int_{a_\delta} \varphi \bar{e}_1 \cdot \bar{n} ds(x) - c \int_{b_\delta} \varphi ds(x)$$

where $a_\delta \equiv \partial B \setminus R_\delta$ and b_δ is the right portion of ∂R_δ , perpendicular to the crack. See Figure A.1! Now, since (according to (ϵ) and the fact that φ is regular on $b_\delta \times [0, T]$ for every $\delta > 0$)

$$\int_{B_\delta} \varphi' da(x) \rightarrow \int_B \varphi' da(x), \quad c \int_{a_\delta} \varphi \bar{e}_1 \cdot \bar{n} ds(x) \rightarrow c \int_{\partial B} \varphi \bar{e}_1 \cdot \bar{n} ds(x) \quad \text{and}$$

$$c \int_{b_\delta} \varphi da(x) \rightarrow 0$$

all uniformly in $[0, T]$ as $\delta \rightarrow 0$, we have

$$\frac{d}{dt} \int_{B_\delta} \varphi da(x) \rightarrow \int_B \varphi' da(x) - c \int_{\partial B} \varphi \bar{e}_1 \cdot \bar{n} ds(x)$$

uniformly in $[0, T]$ as $\delta \rightarrow 0$. From (γ) we have

$$\int_{B_\delta} \varphi da(x) \rightarrow \int_B \varphi da(x)$$

as $\delta \rightarrow 0$ and this proves the Lemma. $\square$

A generalization of Lemma A.3 is given by

Lemma A.4 Let $\varphi(\cdot, t)$, $\chi(\cdot, t)$, $\omega(\cdot, t) : B \searrow C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ be families of fields with φ satisfying (β) , ω satisfying (α) ,

(ζ) $\varphi, \chi \in L^1(B)$ and $\omega|_{\partial B} \in L^1(\partial B)$, all uniformly in time

and

(η) $\dot{\varphi} = \chi + \partial_c \omega$

Then the function (A.2) is C^1 and

$$\frac{d}{dt} \int_B \varphi \, da(x) = \int_B \chi \, da(x) + c \int_{\partial B} \omega \bar{e}_1 \cdot \bar{n} \, ds(x) \quad (A.5)$$

Proof: Analogous to the proof of Lemma A.3 (cf. Gurtin & Yatomi [5], Lemma 2). $\square$

Lemma A.5 Let $\varphi(\cdot, t) : B \searrow C_t \rightarrow \mathbb{R}, \mathbb{R}^2$ be a field satisfying (α) and

(θ) $\frac{\partial \varphi}{\partial x_1}(\cdot, t) \in L^1(B)$ and $\varphi(\cdot, t)|_{\partial B} \in L^1(\partial B)$

Then

$$\int_{\partial B} \varphi \bar{e}_1 \cdot \bar{n} \, ds(x) = \int_B \frac{\partial \varphi}{\partial x_1} \, da(x) \quad (A.6)$$

Proof: As in the proof of Lemma A.3 we find that

$$\int_{a_\delta} \varphi \bar{e}_1 \cdot \bar{n} \, ds(x) - \int_{b_\delta} \varphi \, ds(x) = \int_{B_\delta} \frac{\partial \varphi}{\partial x_1} \, da(x)$$

and by taking limits (as $\delta \rightarrow 0$) formula (A.6) is obtained. $\square$

Lemma A.6 Let $\kappa(\cdot, t) : B \searrow C_t \rightarrow \mathbb{R}^2, \text{End}(\mathbb{R}^2)$ be a field satisfying (α) and

(1) limits $(\kappa)^+$, $(\kappa)^-$, in the sense of (2.6), exist for $q \in [0, \ell(t)[$ and $(\kappa)^+(\cdot, t) \bar{e}_2, (\kappa)^-(\cdot, t) \bar{e}_2 \in L^1([0, \ell(t)[$)

and

(κ) $\text{div } \kappa \in L^1(B)$

Then

$$\lim_{R \rightarrow 0} \int_{O_R(t)} \kappa \bar{e}_R ds(x) = \int_{\partial B} \kappa \bar{n} ds(x) - \int_0^{\ell(t)} ((\kappa)^+ - (\kappa)^-) \bar{e}_2 dq - \int_B \operatorname{div} \kappa da(x) \quad (A.7)$$

where $O_R(t) \equiv \{x \in \mathbb{R}^2 \mid (x_1 - \ell(t))^2 + x_2^2 = R^2\}$ and $\bar{e}_R = \bar{e}_R(t)$ is the outward unit normal to the circle $O_R(t)$.

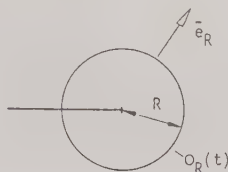


Figure A.2

Proof: An application of the divergence theorem to the field κ and the region $B \setminus \mathcal{D}_R(t)$, where $\mathcal{D}_R(t) \equiv \{x \in \mathbb{R}^2 \mid (x_1 - \ell(t))^2 + x_2^2 \leq R^2\}$, gives

$$\int_{\partial B} \kappa \bar{n} ds(x) - \int_0^{\ell(t)-R} ((\kappa)^+ - (\kappa)^-) \bar{e}_2 dq - \int_{O_R(t)} \kappa \bar{e}_R ds(x) = \int_{B \setminus \mathcal{D}_R(t)} \operatorname{div} \kappa da(x)$$

and by taking limits (as $\delta \rightarrow 0$) we obtain formula (A.7). $\square$

Lemma A.7 Let $\varphi(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$ be a field satisfying (α) , (θ) and (λ) limits $(\varphi)^+$, $(\varphi)^-$, in the sense of (2.6), exist for $q \in [0, \ell(t)[$.

Then

$$\lim_{R \rightarrow 0} \int_{O_R(t)} \varphi \bar{e}_1 \cdot \bar{e}_R ds(x) = 0 \quad (A.8)$$

Proof: Take $\kappa = \varphi \otimes \bar{e}_1$ in Lemma A.6 and then use Lemma A.5. $\square$

Lemma A.8 Let $\varphi(\cdot, t) : B \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2$, $t \in [0, T]$ be a family of fields satisfying conditions (α) , (β) in section 3, (γ) in Lemma A.2, (ε) in Lemma A.3 and in addition

$$(\mu) \quad \frac{\partial \varphi}{\partial x_1}(\cdot, t) \in L^1(B)$$

Then the function (A.2) is C^1 and formula (A.3) is valid.

Proof: From Lemma A.3 and Lemma A.5 we get

$$\frac{d}{dt} \int_B \varphi \, da(x) = \int_B \varphi' \, da(x) - c \int_{\partial B} \varphi \bar{e}_1 \cdot \bar{n} \, ds(x) = \int_B \varphi' \, da(x) - c \int_B \frac{\partial \varphi}{\partial x_1} \, da(x) =$$

$$\int_B (\varphi' - \partial_c \varphi) \, da(x) = \int_B \dot{\varphi} \, da(x)$$

□

Lemma A.9 Let $\psi(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}, \mathbb{R}^2$ be a family of fields satisfying

(v) The mapping $\Omega \ni (q, t) \mapsto \psi(q, t)$, where $\Omega = \{(q, t) \mid 0 \leq q \leq \ell(t), 0 \leq t \leq T\}$, is continuous.

(ξ) If I is an open subinterval of $[0, T]$ and J an open interval satisfying $J \subset [0, \ell(t)[$ for every $t \in I$ then the mapping:
 $J \times I \ni (q, t) \mapsto \psi(q, t)$ is C^1 .

(o) $\dot{\psi} \in L^1(c)$ uniformly in time.

Then the function

$$[0, T] \ni t \mapsto \int_0^{\ell(t)} \psi(q, t) \, dq \tag{A.9}$$

is C^1 and

$$\frac{d}{dt} \int_0^{\ell(t)} \psi \, dq = \int_0^{\ell(t)} \dot{\psi} \, dq + \psi(\ell(t), t) c(t) \tag{A.10}$$

Proof: For $\delta > 0$ we have

$$\frac{d}{dt} \int_0^{\ell(t)-\delta} \psi \, dq = \int_0^{\ell(t)-\delta} \dot{\psi} \, dq + \psi(\ell(t)-\delta, t) c(t).$$

since ψ is regular on $[0, \ell(t)-\delta]$. Then from (v) and (o) it follows that

$$\frac{d}{dt} \int_0^{\ell(t)-\delta} \psi \, dq \rightarrow \int_0^{\ell(t)} \dot{\psi} \, dq + \psi(\ell(t), t) c(t)$$

uniformly in $[0, T]$ as $\delta \rightarrow 0$. Moreover

$$\int_0^{\ell(t)-\delta} \psi \, dq \rightarrow \int_0^{\ell(t)} \psi \, dq$$

as $\delta \rightarrow 0$ and the Lemma now follows from standard theorems on uniform convergence. $\square$

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PAPER III

THE QUASI-STATIC ENERGY RELEASE RATE

AND

THE J-INTEGRAL

IN

THE LINEARIZED THEORY OF ELASTICITY

by

Per Lidström

ABSTRACT

The aim of the present study is to give a rigorous approach to the J-integral method in linear elastic fracture mechanics.

1. INTRODUCTION

The usefulness of the path-independent J-integral in problems concerning the onset of fracture is well established. Its physical basis in the balance of energy for the fracturing body is equally well known. However, the problem of specifying appropriate mathematical assumptions for a rigorous derivation of the J-integral fracture criterion has only recently attracted attention (see Gurtin [1]).

It is the purpose of this paper to present a rigorous approach to the J-integral in the linearized theory of elasticity. We discuss a boundary-initial-value problem for a cylindrical body containing a plane crack extending in its plane. We assume quasi-static behaviour (inertia effects are neglected) and confine our analysis to plane strain (mode I) and anti-plane shear (mode III) deformations. By using an elegant technique presented in Knowles & Pucik [2] we are able to prove the J-integral fracture criterion under natural and rather weak assumptions, at least in the case of anti-plane shear. In plane strain we need an additional steady state condition.

Notation: The analysis takes place in a three-dimensional Euclidean space $\mathbb{R}^3$. Let $\mathbb{R}^2$ denote a two-dimensional subspace. Points in $\mathbb{R}^3$ are denoted: $x, y, \dots$, vectors: $\bar{a}, \bar{b}, \dots$, and tensors ("linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$ or from $\mathbb{R}^2$ into $\mathbb{R}^2$ "): $\underline{A}, \underline{B}, \dots$; $\underline{1}$ is the unit tensor, $\underline{A}^T$ is the transpose of $\underline{A}$, $\det \underline{A}$ the determinant of $\underline{A}$ and $\text{tr } \underline{A}$ the trace of $\underline{A}$. In standard indicial notation and rectangular Cartesian coordinates we have (the usual summation convention is employed)

$$\bar{a} \cdot \bar{b} = a_i b_i, \quad \underline{A} \cdot \underline{B} = A_{ij} B_{ij}$$

$$\bar{a} \otimes \bar{b} \text{ is a tensor with components } a_i b_j$$

$$\text{div } \underline{T} \text{ is a vector with components } \partial T_{ij} / \partial x_j$$

$$\nabla \bar{u} \text{ is a tensor with components } \partial u_i / \partial x_j$$

$$\text{End}(\mathbb{R}^3) = \{\text{tensors in } \mathbb{R}^3\}$$

$$\text{Sym}(\mathbb{R}^3) = \{\underline{A} \in \text{End}(\mathbb{R}^3) \mid \underline{A}^T = \underline{A}\}$$

$$C^m(\mathcal{D}) = \{m\text{-continuously differentiable functions on } \mathcal{D}\}$$

da denotes (Lebesgue) area measure in $\mathbb{R}^2$ and ds denotes arc-length measure.

$$L^p(\mathcal{D}) = \{p\text{-integrable functions on } \mathcal{D}\}$$

2. BASIC EQUATIONS

A body is identified with a reference configuration B . Let $(\bar{e}_1 \bar{e}_2 \bar{e}_3)_0$, $x = (x_1, x_2, x_3)$ be a rectangular Cartesian coordinate system. Let B be a cylindrical region with generators parallel to the x_3 -axis and with end-faces at $x_3 = \pm h$ ($h > 0$).

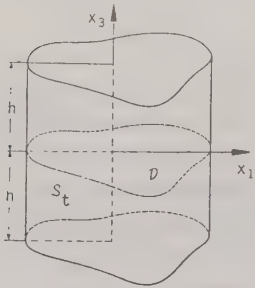
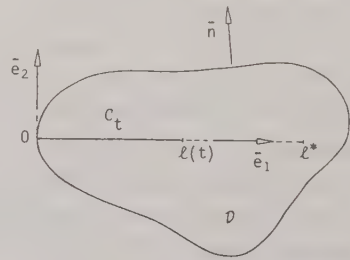


Figure 2.1



The plane cross-section

$$D \equiv \{x \in B \mid x_3 = 0\} \quad (2.1)$$

is assumed to be a bounded, regular region in $\mathbb{R}^2$. We consider a fixed time-interval $[0, T]$ and we assume that B contains a plane edge-crack S_t extending in its plane

$$S_t \equiv \{x \in B \mid 0 \leq x_1 \leq l(t), \quad x_2 = 0\}$$

where the function

$$l : [0, T] \rightarrow \mathbb{R}^+ \quad (2.2)$$

represents the length of the crack. Put

$$l^* \equiv \sup_{t \in [0, T]} l(t) \quad (2.3)$$

$$C^* \equiv \{x \in \mathbb{R}^3 \mid 0 \leq x_1 \leq l^*, \quad x_2 = x_3 = 0\}$$

and

$$C_t = \{x \in \mathbb{R}^3 \mid 0 \leq x_1 \leq \ell(t), \quad x_2 = x_3 = 0\} \quad (2.4)$$

i.e. the line segment of the crack situated in the plane cross-section $\mathcal{D}$.
Let

$$\bar{u}(\cdot, t) : B \setminus S_t \rightarrow \mathbb{R}^3$$

denote the displacement of the body at time $t \in [0, T]$. We assume that for each t , the mapping

$$B \setminus S_t \ni x \mapsto x + \bar{u}(x, t)$$

is smooth and one-to-one onto its image and with smooth inverse, and that

$$\det[\mathbb{I} + \nabla \bar{u}(x, t)] > 0, \quad x \in B \setminus S_t, \quad t \in [0, T]$$

Let

$$\mathbb{I}(\cdot, t) : B \setminus S_t \rightarrow \text{End}(\mathbb{R}^3)$$

denote the (Piola-Kirchhoff) stress tensor. The body under consideration is assumed to be homogeneous and to consist of isotropic elastic material under isothermal conditions. In this paper we confine our attention to the linearized theory where we have

$$\mathbb{I} = \lambda (\text{tr } \underline{\underline{E}}) \mathbb{I} + 2\mu \underline{\underline{E}} \quad (2.5)$$

where

$$\underline{\underline{E}}(x, t) \equiv \frac{1}{2} (\nabla \bar{u}(x, t) + \nabla \bar{u}(x, t)^T)^T \quad ^1$$

is the infinitesimal strain tensor and λ, μ are the Lamé moduli satisfying:

$$\mu > 0, \quad 2\mu + 3\lambda > 0 \quad (2.6)$$

The crack-surface material is characterized by the surface-energy σ (energy per unit area in the reference configuration), a material constant representing the energy-content of the crack-surface. In our simpli-

¹ $\nabla \equiv \partial/\partial x$

fied model we assume that the surface-energy σ is independent of the deformation of the crack-surface, thus putting surface stress equal to zero.

We now specialize to

Plane strain: A deformation $\bar{u}$ is called a plain strain deformation of B if

$$\bar{u} = \bar{e}_1 u_1 + \bar{e}_2 u_2 \quad (2.7)$$

where the "in-plane" displacements

$$u_\alpha(\cdot, t) : \mathcal{D} \setminus C_t \rightarrow \mathbb{R}, \quad \alpha = 1, 2$$

are smooth functions. The plane strain displacement field is independent of the x_3 -coordinate

$$\bar{u}(x, t) = \bar{u}(x_1, x_2, x_3, t) = \bar{e}_1 u_1(x_1, x_2, t) + \bar{e}_2 u_2(x_1, x_2, t)$$

It is a simple matter to confirm that $(\nabla u_\alpha = \bar{e}_1(\partial u_\alpha / \partial x_1) + \bar{e}_2(\partial u_\alpha / \partial x_2))$

$$\nabla \bar{u} = \bar{e}_1 \otimes \nabla u_1 + \bar{e}_2 \otimes \nabla u_2$$

$$\underline{E} = \frac{1}{2} (\bar{e}_1 \otimes \nabla u_1 + \nabla u_1 \otimes \bar{e}_1 + \bar{e}_2 \otimes \nabla u_2 + \nabla u_2 \otimes \bar{e}_2)$$

$$\underline{I} = \underline{I}_p + \bar{e}_3 \otimes \bar{e}_3 \lambda \operatorname{tr} \underline{E} \quad (2.8)$$

$$\underline{I}_p = \lambda (\operatorname{tr} \underline{E}) \underline{I}_p + 2 \mu \underline{E}$$

where

$$\underline{I}_p = \bar{e}_1 \otimes \bar{e}_1 + \bar{e}_2 \otimes \bar{e}_2 = \underline{I} - \bar{e}_3 \otimes \bar{e}_3$$

is the unit tensor in $\mathbb{R}^2$.

The displacement of the crack at time t is given by $(x = (x_1, x_2) \in \mathcal{D})$:

$$u_\alpha^+(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}, \quad u_\alpha^-(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}$$

where

$$u_{\alpha}^{+}(q, t) \equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 > 0}} u_{\alpha}(x, t), \quad u_{\alpha}^{-}(q, t) \equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 < 0}} u_{\alpha}(x, t), \quad q \in [0, \ell(t)[$$

$$(2.9)$$

$$u_{\alpha}^{+}(\ell(t), t) = u_{\alpha}^{-}(\ell(t), t) \equiv \lim_{\substack{x \rightarrow (\ell(t), 0) \\ x \in \mathcal{D} \setminus C_t}} u_{\alpha}(x, t) \quad (2.10)$$

Denote by D_p the set of plane strain displacement fields satisfying the items (i)_p - (v)_p stated below:

$$(i)_p \quad \ell \in C^1([0, T]), \quad \ell(t) > 0 \text{ for } t \in [0, T] \text{ and } C^* \subset \bar{\mathcal{D}}.$$

$$(ii)_p \quad u_{\alpha}(\cdot, t) \in C^2(\mathcal{D} \setminus C_t).$$

$$(iii)_p \quad \text{Limits } (\nabla u_{\alpha})^{+}, (\nabla u_{\alpha})^{-}, \text{ in the sense of (2.9), exist for } q \in [0, \ell(t)[.$$

$$(iv)_p \quad \text{If } P \text{ is an open subset of } \mathcal{D} \text{ and } I \text{ an open subinterval in } [0, T] \text{ satisfying } P \cap C_t = \emptyset \text{ for every } t \in I \text{ then the mappings}$$

$$P \times I \ni (x, t) \leadsto u_{\alpha}(x, t)$$

are C^2 .

The material velocity $\dot{\bar{u}}$ is defined by

$$\dot{\bar{u}}(x, t) \equiv \bar{e}_1 \dot{u}_1(x, t) + \bar{e}_2 \dot{u}_2(x, t)$$

$$\dot{u}_{\alpha}(x, t) \equiv (\partial u_{\alpha} / \partial t)(x, t) \quad x \in \mathcal{D} \setminus C_t$$

$$(v)_p \quad \bar{E}(\cdot, t) \in L^2(\mathcal{D}) \text{ for } t \in [0, T] \text{ and the function:}$$

$$[0, T] \ni t \leadsto \int_{\mathcal{D}} \left(\frac{\lambda}{2} (\text{tr } \bar{E}(x, t))^2 + \mu |\bar{E}(x, t)|^2 \right) da(x) \quad (2.11)$$

representing the strain energy of the body (per unit length of the crack-front), is C^1 .

Remark: Note that $|\text{tr } \bar{E}| \leq \sqrt{2} |\bar{E}|$ and thus the strain energy density

$$\frac{\lambda}{2} (\text{tr } \bar{E})^2 + \mu |\bar{E}|^2 \leq (\lambda + \mu) |\bar{E}|^2$$

$$^1 |\underline{A}| \equiv \sqrt{\underline{A} \cdot \underline{A}}$$

The crack-tip speed (in the reference configuration)

$$c : [0, T] \rightarrow \mathbb{R}$$

is defined by

$$c(t) \equiv \frac{d\ell}{dt}(t)$$

We now state a boundary-initial-value problem for the fracturing, elastic body B undergoing quasi-static, plane strain:

Find $\bar{u} \in D_p$ so that

$$\operatorname{div} \bar{T}_p = \bar{0}, \quad x \in \mathcal{D} \setminus C_t \quad (I)_p$$

$$\bar{T}_p = \lambda (\operatorname{tr} \bar{E}) \bar{1}_p + 2 \mu \bar{E}$$

$$(\bar{T}_p)^+ \bar{e}_2 = \bar{0}$$

$$(\bar{T}_p)^- \bar{e}_2 = \bar{0} \quad q \in [0, \ell(t)[\quad (II)_p$$

$$\bar{0} = \int_{\partial \mathcal{D}} \bar{T}_p \bar{n} \, ds(x) \quad (III)_p$$

where $\bar{n}$ denotes the outward unit normal on $\partial \mathcal{D}$ (see Figure 2.1).

$$\bar{0} = \int_{\partial \mathcal{D}} (x - 0) \times \bar{T}_p \bar{n} \, ds(x) \quad (IV)_p$$

$$2 \sigma c = \int_{\partial \mathcal{D}} \dot{\bar{u}} \cdot \bar{T}_p \bar{n} \, ds(x) - \frac{d}{dt} \int_{\mathcal{D}} w \, da(x) \quad (V)_p$$

where $w(x, t) \equiv \frac{\lambda}{2} (\operatorname{tr} \bar{E}(x, t))^2 + \mu |\bar{E}(x, t)|^2$ (the strain-energy density).

$$\bar{T}_p \bar{n} = \bar{t}_0, \quad x \in \partial \mathcal{D} \quad (VI)_p$$

where the boundary traction $\bar{t}_0 : \partial \mathcal{D} \rightarrow \mathbb{R}^2$ is a prescribed,

time-independent, continuous field satisfying

$$\int_{\partial D} \bar{t}_0 \, ds(x) = \bar{0}, \quad \int_{\partial D} (x - 0) \times \bar{t}_0 \, ds(x) = \bar{0} \quad (2.12)$$

$$\ell(0) = \ell_0$$

$$\begin{aligned} \bar{u}(x, 0) &= \bar{0} \\ \dot{\bar{u}}(x, 0) &= \bar{0} \end{aligned} \quad x \in D \setminus C_{t=0} \quad (VII)_p$$

where $\ell_0 > 0$.

Remark 1: Condition $(I)_p$ is equivalent to (Navier's equation)

$$\mu \Delta \bar{u} + (\lambda + \mu) \nabla \operatorname{div} \bar{u} = \bar{0}$$

Remark 2: In this paper we limit our discussion to quasi-static deformations (inertia effects are neglected) in the absence of body forces and with stress-free crack-surfaces. For a boundary-initial-value problem without these restrictions see Lidström [3].

Remark 3: The relation of conditions $(I)_p - (VI)_p$ to the global balance of momentum, moment of momentum and energy for the fracturing body B is discussed in Lidström [4]. The right membrum of equation $(V)_p$ is called the energy release rate for B at time t (per unit length of the crack-front):

$$E(D, t) \equiv \int_{\partial D} \dot{\bar{u}} \cdot \bar{T}_p \bar{n} \, ds(x) - \frac{d}{dt} \int_D w \, da(x)$$

Note that if $\bar{u}$ is a solution to the boundary-initial-value problem then we have

$$E(D', t) = E(D, t)$$

for any regular subregion D' of D containing the crack-tip. This follows from the divergence theorem, $(I)_p$ and the relation

$$\dot{w} = \bar{T}_p \cdot \dot{\bar{E}}$$

Analogously the right membra of equations $(III)_p$ and $(IV)_p$ may be called

the momentum release rate $M(\mathcal{D}, t)$ and the moment of momentum release rate $mM(\mathcal{D}, t)$ respectively. It is a simple matter to confirm that

$$M(\mathcal{D}', t) = M(\mathcal{D}, t), \quad mM(\mathcal{D}', t) = mM(\mathcal{D}, t)$$

Remark 4: The equations $(I)_P - (VII)_P$ constitute a two-dimensional boundary-initial-value problem. In the three-dimensional context they may be supplemented by the boundary-conditions

$$\mathbb{I} \bar{e}_3 = \bar{0}, \quad x_3 = \pm h$$

representing traction-free end-faces of the cylinder. Note that

$$\mathbb{I} \bar{e}_3 = \bar{e}_3(\lambda \operatorname{tr} \underline{E}) = \bar{e}_3(\nu \operatorname{tr} \mathbb{I}_P)$$

where ν is Poisson's ratio: $\nu = \lambda / (2(\lambda + \mu))$.

We also consider

Anti-plane shear: A deformation $\bar{u}$ is called an anti-plane shear deformation of B if

$$\bar{u} = \bar{e}_3 u \tag{2.13}$$

where the "out-of-plane" displacement

$$u(\cdot, t) : \mathcal{D} \setminus C_t \rightarrow \mathbb{R}$$

is a smooth function. The anti-plane shear displacement field is independent of the x_3 -coordinate

$$\bar{u}(x, t) = \bar{u}(x_1, x_2, x_3, t) = \bar{e}_3 u(x_1, x_2, t)$$

It is a simple matter to confirm that

$$\nabla \bar{u} = \bar{e}_3 \otimes \nabla u$$

$$\underline{E} = \frac{1}{2} (\bar{e}_3 \otimes \nabla u + \nabla u \otimes \bar{e}_3) \tag{2.14}$$

$$\operatorname{tr} \underline{E} = 0$$

$$\underline{I} = \mu (\bar{e}_3 \otimes \nabla u + \nabla u \otimes \bar{e}_3) \quad (2.14)_4$$

Remark: Note that $\det(\underline{I} + \nabla \bar{u}) = 1$ and thus the anti-plane shear deformation is isochoric.

The displacement of the crack at time t is given by:

$$u^+(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}, \quad u^-(\cdot, t) : [0, \ell(t)] \rightarrow \mathbb{R}$$

and these functions are defined by substituting u_α for u in (2.9) and (2.10).

Denote by D_a the set of anti-plane shear displacement fields satisfying the items (i)_a - (v)_a stated below:

$$(i)_a = (i)_p$$

$$(ii)_a = (ii)_p$$

$$(iii)_a = (iii)_p$$

$$(iv)_a = (iv)_p$$

where u_α in (i)_p - (iv)_p is substituted for u .

(v)_a $\nabla u(\cdot, t) \in L^2(\mathcal{D})$ for $t \in [0, T]$ and the function:

$$[0, T] \ni t \leadsto \int_{\mathcal{D}} \frac{\mu}{2} |\nabla u(x, t)|^2 da(x) \quad (2.15)$$

representing the strain-energy of the body (per unit length of the crack-front), is C^1 .

We now state a boundary-initial-value problem for the fracturing, elastic body B undergoing quasi-static, anti-plane shear:

Find $u \in D_a$ so that

$$\Delta u = 0, \quad x \in \mathcal{D} \setminus C_t \quad (I)_a$$

$$\begin{aligned} \left(\frac{\partial u}{\partial x_2}\right)^+ &= 0 \\ \left(\frac{\partial u}{\partial x_2}\right)^- &= 0 \end{aligned} \quad q \in [0, \ell(t)] \quad (II)_a$$

¹ $|\tilde{a}| \equiv \sqrt{\tilde{a} \cdot \tilde{a}}$

$$0 = \int_{\partial\mathcal{D}} \bar{\mathbf{n}} \cdot \nabla u \, ds(x) \quad (\text{III})_a$$

$$0 = \int_{\partial\mathcal{D}} (x - 0) \bar{\mathbf{n}} \cdot \nabla u \, ds(x) \quad (\text{IV})_a$$

$$2\sigma c = \int_{\partial\mathcal{D}} \mu \dot{u} \bar{\mathbf{n}} \cdot \nabla u \, ds(x) - \frac{d}{dt} \int_{\mathcal{D}} \frac{\mu}{2} |\nabla u|^2 \, da(x) \quad (\text{V})_a$$

$$\mu \bar{\mathbf{n}} \cdot \nabla u = t_0, \quad x \in \partial\mathcal{D} \quad (\text{VI})_a$$

where the boundary traction $t_0: \partial\mathcal{D} \rightarrow \mathbb{R}$ is a prescribed, time-independent, continuous field satisfying

$$\int_{\partial\mathcal{D}} t_0 \, ds(x) = 0, \quad \int_{\partial\mathcal{D}} (x - 0) t_0 \, ds(x) = \bar{0} \quad (2.16)$$

$$\ell(0) = \ell_0$$

$$u(x, 0) = 0 \quad (\text{VII})_a$$

$$\dot{u}(x, t) = 0 \quad x \in \mathcal{D} \setminus C_{t=0}$$

where $\ell_0 > 0$.

Remark: In anti-plane shear the end-face traction field $\bar{\mathbf{f}}$ (at $x_3 = h$) is determined by

$$\bar{\mathbf{f}} = \mathbb{I} \bar{\mathbf{e}}_3 = \mu \nabla u$$

We close this section with the following important observation: Condition (2.12) (and (2.16) in the case of anti-plane shear), representing the equilibrium of external forces, is a necessary condition for the existence of a solution to the boundary-initial-value problem. On the other hand, if (2.12) (or (2.16)) is fulfilled then, from (VI)_p (or (VI)_a in the case of anti-plane shear), it follows that (III)_p and (IV)_p (or (III)_a, (IV)_a) are satisfied automatically.

3. SANDERS' I-INTEGRAL

In this section we assume that $\bar{u}$ is a plane strain solution and u an anti-plane shear solution to the boundary-initial-value problems stated in the previous section. Furthermore we assume that for $t \in [0, T]$

$$\begin{aligned} (vi)_p \quad \sup_{x \in \mathcal{D} \setminus C_t} |\bar{u}(x, t)| &< \infty \\ (vi)_a \quad \sup_{x \in \mathcal{D} \setminus C_t} |u(x, t)| &< \infty \end{aligned} \quad (3.1)$$

Proposition 1 We have, in plane strain

$$\frac{d}{dt} \int_{\mathcal{D}} w \, da(x) = \frac{1}{2} \int_{\partial \mathcal{D}} \bar{t}_0 \cdot \dot{\bar{u}} \, ds(x) \quad (3.2)$$

and in anti-plane shear

$$\frac{d}{dt} \int_{\mathcal{D}} \frac{\mu}{2} |\nabla u|^2 \, da(x) = \frac{1}{2} \int_{\partial \mathcal{D}} t_0 \cdot \dot{u} \, ds(x) \quad (3.3)$$

Proof: For fixed t take $\underline{A} = \underline{I}(\cdot, t)$ and $\bar{v} = \bar{u}(\cdot, t)$ in Theorem A.1 (see Appendix!). Since, according to (3.1), $\bar{u}$ is bounded and since we have the estimate

$$\underline{I} \cdot \nabla \bar{u} = \underline{I} \cdot \underline{E} \geq \min\left(\frac{1}{2\mu}, \frac{1}{3\lambda + 2\mu}\right) |\underline{I}|^2, \quad x \in \mathcal{D} \setminus C_t$$

we may conclude that $\underline{I} \cdot \nabla \bar{u} \in L^1(\mathcal{D})$ and that

$$\int_{\mathcal{D}} \underline{I} \cdot \nabla \bar{u} \, da(x) = \int_{\partial \mathcal{D}} \bar{u} \cdot \underline{I} \bar{n} \, ds(x)$$

In plane strain we have: $\underline{I} \cdot \nabla \bar{u} = \underline{I}_p \cdot \underline{E} = 2w$ in $\mathcal{D} \setminus C_t$ and $\underline{I} \bar{n} = \underline{I}_p \bar{n} = \bar{t}_0$ on $\partial \mathcal{D}$. Thus

$$2 \int_{\mathcal{D}} w \, da(x) = \int_{\partial \mathcal{D}} \bar{t}_0 \cdot \bar{u} \, ds(x) \quad (3.4)$$

Since $\bar{t}_0$ is time-independent (3.2) follows by differentiating (3.4).

In anti-plane shear we have: $\underline{I} \cdot \nabla u = \mu |\nabla u|^2$ in $\mathcal{D} \setminus C_t$ and $\underline{I} \bar{n} = \bar{e}_3 \mu \bar{n} \cdot \nabla u = \bar{e}_3 t_0$ on $\partial \mathcal{D}$. $\square$

Remark 1: From the proof of Proposition 1 it follows that it is possible to omit item (v)_p in the definition of D_p, since the differentiability of the function (2.11) is a consequence of (i)_p - (iv)_p, (vi)_p, (I)_p, (II)_p and (VI)_p.

Remark 2: Proposition 1 asserts that the working of the surface tractions equals twice the rate of change of strain energy. It follows that (V)_p is equivalent to

$$2\sigma c = \frac{1}{2} \int_{\partial D} \bar{t}_0 \cdot \dot{\bar{u}} \, ds(x)$$

and thus, when the crack (with $\sigma > 0$) extends ($c > 0$) under constant load, one half of the work done by the loading forces is used to increase the strain energy of the body and the other half is used for the crack-propagation (cf. Bueckner [5]).

Denote by D_{ps} the set of displacement fields $\bar{u} \in D_p$ satisfying (vi)_p, stated above, and

(vii)_p The limits $(\dot{\bar{u}})^+$, $(\dot{\bar{u}})^-$, $(\nabla \dot{\bar{u}})^+$ and $(\nabla \dot{\bar{u}})^-$, in the sense of (2.9), exist for $q \in [0, \ell(t)[$ and

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla \bar{u})^\pm(q, t) &= (\nabla \dot{\bar{u}})^\pm(q, t) \\ q &\in [0, \ell(t)[\\ \frac{\partial}{\partial q}(\nabla \bar{u})^\pm(q, t) &= \left(\frac{\partial}{\partial x_1} \nabla \bar{u}\right)^\pm(q, t) \end{aligned}$$

The set D_{as} is analogously defined with

(vii)_a The limits $(\dot{u})^+$, $(\dot{u})^-$, $(\nabla \dot{u})^+$ and $(\nabla \dot{u})^-$, in the sense of (2.9), exist for $q \in [0, \ell(t)[$ and

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla u)^\pm(q, t) &= (\nabla \dot{u})^\pm(q, t) \\ q &\in [0, \ell(t)[\\ \frac{\partial}{\partial q}(\nabla u)^\pm(q, t) &= \left(\frac{\partial}{\partial x_1} \nabla u\right)^\pm(q, t) \end{aligned}$$

Now let $\bar{u} \in D_{ps}$ be a plane strain solution and $u \in D_{as}$ an anti-plane shear solution to the boundary-initial-value problems stated in the previous section. Then we have

Proposition 2 In plane strain, condition $(V)_p$ is equivalent to

$$2\sigma c = \frac{1}{2} \int_{\Gamma} (\dot{\bar{u}} \cdot \bar{\Gamma}_p \bar{n} - \bar{u} \cdot \dot{\bar{\Gamma}}_p \bar{n}) ds(x) \quad (3.5)$$

and in anti-plane shear condition $(V)_a$ is equivalent to

$$2\sigma c = \frac{1}{2} \int_{\Gamma} (\dot{\bar{u}} \bar{n} \cdot \nabla u - \bar{u} \bar{n} \cdot \nabla \dot{\bar{u}}) ds(x) \quad (3.6)$$

where $\Gamma \equiv \partial \mathcal{D}'$ and $\mathcal{D}'$ is an arbitrary regular subregion of $\mathcal{D}$ containing the crack-tip.

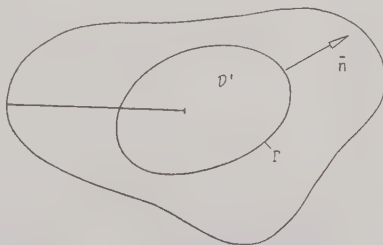


Figure 3.1

Proof: (Plane strain): Since $\dot{\bar{\Gamma}}_p \bar{n} = \dot{\bar{t}}_0 = \bar{0}$ on $\partial \mathcal{D}$ we have (from Proposition 1)

$$2\sigma c = \frac{1}{2} \int_{\partial \mathcal{D}} (\dot{\bar{u}} \cdot \bar{\Gamma}_p \bar{n} - \bar{u} \cdot \dot{\bar{\Gamma}}_p \bar{n}) ds(x)$$

From $(I)_p$ and a simple calculation we get

$$\operatorname{div}(\bar{\Gamma}_p^T \dot{\bar{u}} - \dot{\bar{\Gamma}}_p^T \bar{u}) = 0, \quad x \in \mathcal{D} \setminus \mathcal{C}_t$$

and from $(II)_p$ and $(vii)_p$ we get

$$(\dot{\bar{\Gamma}}_p)^{\pm} \bar{e}_2 = \bar{0}, \quad q \in [0, \ell(t)[$$

Then by applying the divergence theorem in $\mathbb{R}^2$ to the region $\mathcal{D} \setminus (\mathcal{D}' \cup \mathcal{C}_t)$ and the field $\bar{\Gamma}_p^T \dot{\bar{u}} - \dot{\bar{\Gamma}}_p^T \bar{u}$ we obtain

$$\int_{\partial \mathcal{D}} (\dot{\bar{u}} \cdot \bar{\Gamma}_p \bar{n} - \bar{u} \cdot \dot{\bar{\Gamma}}_p \bar{n}) ds(x) = \int_{\Gamma} (\dot{\bar{u}} \cdot \bar{\Gamma}_p \bar{n} - \bar{u} \cdot \dot{\bar{\Gamma}}_p \bar{n}) ds(x)$$

and this proves (3.5). The anti-plane shear case is equally simple. $\square$

Remark 1: The integral

$$I = I(t) \equiv \frac{1}{2} \int_{\Gamma} (\dot{\bar{u}} \cdot \bar{T}_p \bar{n} - \bar{u} \cdot \dot{\bar{T}}_p \bar{n}) \, ds(x)$$

may be referred to as Sanders' I-integral (cf. Sanders [6]). It is obvious that Γ in Proposition 2 may be replaced by any regular curve in $\mathcal{D}$ which begins and ends on C_t and surrounds the crack-tip. The I-integral is said to be path-independent.

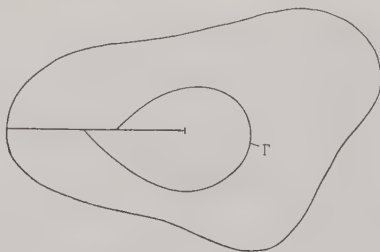


Figure 3.2

Remark 2: It is possible to generalize Propositions 1 and 2 to a fracturing body consisting of anisotropic, linear elastic material defined by a symmetric, positive definite elasticity tensor.

4. RICE'S J-INTEGRAL

We begin this section by introducing some basic definitions. We assume that $\ell \in C^1([0, T])$. Let

$$\varphi(\cdot, t) : \mathcal{D} \setminus C_t \rightarrow \mathbb{R}, \mathbb{R}^2, \dots \quad (4.1)$$

$t \in [0, T]$ be a family of fields satisfying:

- (a) $\varphi(\cdot, t) \in C^1(\mathcal{D} \setminus C_t)$
- (b) If P is an open subset of $\mathcal{D}$ and I an open subinterval in $[0, T]$ satisfying $P \cap C_t = \emptyset$ for every $t \in I$ then the mapping

$$P \times I \ni (x, t) \mapsto \varphi(x, t)$$

is C^1 .

Following Strifors [7] we may then introduce the convected crack-tip time-derivative

$$\varphi'(x, t) \equiv \frac{d}{ds} \varphi(x + \bar{e}_1 c(t)s, t + s) \big|_{s=0}, \quad x \in \mathcal{D} \setminus C_t \quad (4.2)$$

A simple computation gives the important formula

$$\varphi' = \dot{\varphi} + c \frac{\partial \varphi}{\partial x_1} \quad (4.3)$$

Put (for $R_0 > 0$)

$$\begin{aligned} \bar{\ell}(t) &\equiv (\ell(t), 0), \quad \Delta(R_0, t) \equiv \{x \in \mathbb{R}^2 \mid |x - \bar{\ell}(t)| \leq R_0\} \setminus C_t \\ \Omega(R_0) &\equiv \{\bar{r} \in \mathbb{R}^2 \mid \bar{r} = (\bar{e}_1 \cos \theta + \bar{e}_2 \sin \theta) r, r \in]0, R_0], \theta \in]-\pi, \pi[\} \end{aligned} \quad (4.4)$$

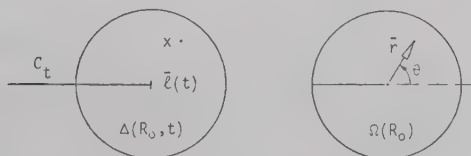


Figure 4.1

Let $I \subset [0, T]$ be a time interval satisfying

$$\Delta(R_0, t) \subset \mathcal{D} \text{ for all } t \in I$$

We carry out a simple change of independent variables according to the mapping

$$\Omega(R_0) \times I \ni (\bar{r}, t) \sim (\bar{\ell}(t) + \bar{r}, t) \in \Delta_I(R_0) \quad (4.5)$$

where $\Delta_I(R_0) \equiv \{(x, t) \mid x \in \Delta(R_0, t), t \in I\}$. Given a field as in (4.1) we may introduce the associated field

$$\tilde{\varphi}: \Omega(R_0) \times I \rightarrow \mathbb{R}, \mathbb{R}^2, \dots \quad (4.6)$$

defined by

$$\tilde{\varphi}(\bar{r}, t) \equiv \varphi(\bar{\ell}(t) + \bar{r}, t)$$

We have

$$\varphi'(x, t) = \frac{\partial \tilde{\varphi}}{\partial t}(x - \bar{\ell}(t), t), \quad x \in \Delta(R_0, t), \quad t \in I \quad (4.7)$$

Remark: We agree to write $\tilde{\varphi}(\bar{r}, t) = \tilde{\varphi}(r, \theta, t)$, $(r, \theta) \in [0, R_0] \times]-\pi, \pi[$ and $(\tilde{\varphi})^\pm$ for the limit functions

$$(\tilde{\varphi})^\pm: [0, R_0] \times I \rightarrow \mathbb{R}, \mathbb{R}^2, \dots$$

defined by

$$(\tilde{\varphi})^+(\rho, t) \equiv \lim_{(r, \theta) \rightarrow (\rho, \pi)} \tilde{\varphi}(r, \theta, t), \quad (\tilde{\varphi})^-(\rho, t) \equiv \lim_{(r, \theta) \rightarrow (\rho, -\pi)} \tilde{\varphi}(r, \theta, t), \quad \rho \in [0, R_0] \quad (4.8)$$

$$(\tilde{\varphi})^+(0, t) = (\tilde{\varphi})^-(0, t) \equiv \lim_{(r, \theta) \rightarrow (0, \theta)} \tilde{\varphi}(r, \theta, t) \quad (4.9)$$

Note that $(\tilde{\varphi})^\pm(\rho, t) = (\varphi)^\pm(\bar{\ell}(t) + \rho, t)$ for $\rho \in [0, R_0]$, $t \in I$.

Denote by D'_{ps} the set of displacement fields $\bar{u} \in D_{ps}$ satisfying:

$$(viii)_p \quad \text{tr } \underline{E}'(x, t) \quad \text{and} \quad \underline{W}'(x, t) = \underline{0} \quad \text{for all } x \in \mathcal{D} \setminus C_t, \quad t \in [0, T]$$

where

$$\underline{W} = \frac{1}{2} (\nabla \bar{u} - \nabla \bar{u}^T)$$

is the (infinitesimal) rotation tensor.

Remark: The requirements introduced in $(viii)_p$ are equivalent to

$$(\text{tr } \underline{E})' = -c \frac{\partial}{\partial x_1} \text{tr } \underline{E}, \quad \dot{\underline{W}} = -c \frac{\partial}{\partial x_1} \underline{W}, \quad x \in \mathcal{D} \setminus C_t$$

and we say that steady state conditions for the fields $\text{tr } \underline{E}$ and $\underline{W}$ are prevailing at the crack-tip. Note that in anti-plane shear we have

$$\text{tr } \underline{E}' = 0, \quad \underline{W}' = \frac{1}{2} (\bar{e}_3 \otimes \nabla u' - \nabla u' \otimes \bar{e}_3)$$

Let $\bar{u} \in D'_{ps}$ be a plane strain solution and $u \in D_{as}$ an anti-plane shear solution to the boundary-initial-value problems stated in section 2. Then we have

Proposition 3 In plane strain

$$\frac{1}{2\pi R} \int_{O_R(t)} |\bar{u}'|^2 ds(x) \quad (4.10)$$

remains bounded as $R \rightarrow 0$, where

$$O_R(t) = \{x \in \mathbb{R}^2 \mid (x_1 - \ell(t))^2 + x_2^2 = R^2\} \quad (4.11)$$

Moreover we have $\underline{I}'_p(\cdot, t) \in L^2(\mathcal{D})$. In anti-plane shear

$$\frac{1}{2\pi R} \int_{O_R(t)} (u')^2 ds(x) \quad (4.12)$$

remains bounded as $R \rightarrow 0$ and $\nabla u'(\cdot, t) \in L^2(\mathcal{D})$.

Proof: Choose $R_0 > 0$ sufficiently small so that $\Delta(R_0, t) \subset \mathcal{D}$ for all $t \in [0, T]$. For fixed $s, t \in]0, T[$, $s \neq t$ we introduce the functions

$$\tilde{A} : \Omega(R_0) \rightarrow \text{End}(\mathbb{R}^3), \quad \tilde{V} : \Omega(R_0) \rightarrow \mathbb{R}^3$$

defined by

$$\begin{aligned} \tilde{A}(\vec{r}, t, s) &\equiv \frac{1}{t-s} (\tilde{I}(\vec{r}, t) - \tilde{I}(\vec{r}, s)) \\ \tilde{V}(\vec{r}, t, s) &\equiv \frac{1}{t-s} (\tilde{U}(\vec{r}, t) - \tilde{U}(\vec{r}, s)) \end{aligned} \quad (4.13)$$

It is a simple matter to confirm that the prerequisites for an application of Theorem A.1 (or rather the reformulation presented in the remark following Theorem A.1) are at hand. For instance we have

$$\tilde{A} \cdot \nabla \tilde{V} \geq \min\left(\frac{1}{2\mu}, \frac{1}{3\lambda + 2\mu}\right) |\tilde{A}|^2, \quad \vec{r} \in \Omega(R_0)$$

(where ∇ denotes the gradient with respect to $\vec{r}$). Thus we have

$$\tilde{A} \cdot \nabla \tilde{V} \in L^1(\Omega(R_0))$$

and for $0 < R \leq R_0$ we have (according to formula (A.7))

$$\begin{aligned} 0 &\leq \int_{-\pi}^{\pi} \int_0^R \tilde{A}(r, \theta, t, s) \cdot \nabla \tilde{V}(r, \theta, t, s) r dr d\theta = \\ &R \int_{-\pi}^{\pi} \tilde{V}(R, \theta, t, s) \cdot \tilde{A}(R, \theta, t, s) \bar{e}_r d\theta \end{aligned} \quad (4.14)$$

(where $\bar{e}_r = \bar{e}_1 \cos \theta + \bar{e}_2 \sin \theta$). From (4.7) and (4.13) we have

$$\lim_{s \rightarrow t} \tilde{V}(R, \theta, t, s) \cdot \tilde{A}(R, \theta, t, s) \bar{e}_r = \tilde{U}'(R, \theta, t) \cdot \tilde{I}'(R, \theta, t) \bar{e}_r$$

Furthermore we have the simple estimate

$$|\tilde{V}(R, \theta, t, s) \cdot \tilde{A}(R, \theta, t, s) \bar{e}_r| \leq |\tilde{V}(R, \theta, t, s)| |\tilde{A}(R, \theta, t, s)| \leq C_R < \infty$$

where the upper bound C_R (independent of θ, t, s possibly depending on R) is given by

$$C_R = \sup_{\substack{\theta \in]-\pi, \pi[\\ t \in [0, T]}} |\tilde{U}'(R, \theta, t)| \sup_{\substack{\theta \in]-\pi, \pi[\\ t \in [0, T]}} |\tilde{I}'(R, \theta, t)|$$

Then by taking limits (as $s \rightarrow t$) in (4.14) and by invoking the Lebesgue dominated convergence theorem ¹ (or its Riemann version) we may conclude that

$$\int_{-\pi}^{\pi} \tilde{\mathbf{u}}'(R, \theta, t) \cdot \tilde{\mathbf{T}}'(R, \theta, t) \bar{\mathbf{e}}_r d\theta \geq 0 \quad (4.15)$$

In plane strain we have, since $\bar{\mathbf{u}} \in D'_{ps}$,

$$\mathbf{T}' = \lambda (\text{tr } \mathbf{E}') \mathbf{1} + 2\mu \mathbf{E}' = 2\mu \nabla \bar{\mathbf{u}}' = \mathbf{T}'_p$$

and thus (4.15) is equivalent to

$$\int_{-\pi}^{\pi} \tilde{\mathbf{u}}' \cdot \nabla \tilde{\mathbf{u}}' \bar{\mathbf{e}}_r d\theta \geq 0 \quad (4.16)$$

Consider, for fixed $t \in]0, T[$, the function

$$]0, R_0] \ni R \mapsto g(R) = \int_{-\pi}^{\pi} |\tilde{\mathbf{u}}'(R, \theta, t)|^2 d\theta$$

It is obvious that $g(R) \geq 0$, $R \in]0, R_0]$ and from (4.16) it follows that

$$\frac{dg}{dR}(R) \geq 0, \quad R \in]0, R_0]$$

Consequently g must be bounded on $]0, R_0]$ and thus

$$\int_{-\pi}^{\pi} |\tilde{\mathbf{u}}'|^2 d\theta = \frac{1}{R} \int_{O_R} |\bar{\mathbf{u}}'|^2 ds(x)$$

remains bounded as $R \rightarrow 0$. This proves the first statement in Proposition 3 (in the case of plane strain). Since we have

$$\begin{aligned} \text{div } \mathbf{T}' &= \bar{\mathbf{0}}, \quad x \in \mathcal{D} \setminus C_t \\ (\mathbf{T}')^{\pm} \bar{\mathbf{e}}_2 &= \bar{\mathbf{0}}, \quad q \in [0, \ell(t)[\end{aligned} \quad (4.17)$$

$$\mathbf{T}' \cdot \nabla \bar{\mathbf{u}}' = \mathbf{T}' \cdot \mathbf{E}' \geq \min\left(\frac{1}{2\mu}, \frac{1}{3\lambda + 2\mu}\right) |\mathbf{T}'|^2, \quad x \in \mathcal{D} \setminus C_t$$

we may take $\bar{\mathbf{A}} = \mathbf{T}'$ and $\bar{\mathbf{v}} = \bar{\mathbf{u}}'$ in Theorem A.1 and it follows that

¹ See Schwartz [8], Corollary on p. 54.

$\underline{I}' \cdot \underline{E}' \in L^1(\mathcal{D})$. From (4.17)₃ it now follows that $\underline{I}'_p \in L^2(\mathcal{D})$ and this proves the Proposition in the case of plane strain for $t \in]0, T[$. The end-points of the time-interval, $t = 0, T$, are handled by regarding one-sided limits. In anti-plane shear we have

$$\bar{u}' = \bar{e}_3 u', \quad \underline{I}' = \mu (\bar{e}_3 \otimes \nabla u' + \nabla u' \otimes \bar{e}_3)$$

and (4.15) is equivalent to

$$\int_{-\pi}^{\pi} \tilde{u}' \bar{e}_r \cdot \nabla \tilde{u}' d\theta \geq 0$$

and it follows, in the same manner as above, that

$$\int_{-\pi}^{\pi} (\tilde{u}')^2 d\theta$$

remains bounded as $R \rightarrow 0$ and that $\underline{I}' \cdot \nabla \bar{u}' = \mu |\nabla u'|^2 \in L^1(\mathcal{D})$. $\square$

Remark: It follows from

$$|\underline{E}'|^2 \leq \max\left(\frac{1}{2\mu}, \frac{1}{3\lambda + 2\mu}\right) \underline{I}' \cdot \underline{E}'$$

and the proof of Proposition 3 that if $\bar{u} \in D'_{ps}$ is a plane strain solution to the boundary-initial-value problem in section 2 then $\underline{E}' \in L^2(\mathcal{D})$.

With presumptions as in Proposition 3 we have

Corollary 1 In plane strain we have $\bar{u}' \in L^2(\mathcal{D})$ and

$$\int_{\Delta(R_0, t)} |\bar{u}'|^2 da(x) \leq C_t R_0^2 \quad (4.18)$$

for some $C_t > 0$. An analogous result holds in anti-plane shear.

Proof: From the proof of Proposition 3 it follows that the function

$$]0, R_0[\ni R \leadsto R g(R) = \int_{-\pi}^{\pi} |\tilde{u}'(R, \theta, t)|^2 R d\theta$$

is integrable and that

$$\int_0^{R_0} \left(\int_{-\pi}^{\pi} |\tilde{u}'|^2 R d\theta \right) dR \leq C_t R_0^2$$

for some constant (possibly depending on t) $C_t > 0$. Since the integrand $|\tilde{u}'|^2 R$ is nonnegative (and continuous on $]0, R_0] \times]-\pi, \pi[$) it follows that $\tilde{u}' \in L^2(\Delta(R_0, t))$ and that (4.18) is valid. $\square$

We now turn to our main result

Theorem 1 If $\tilde{u} \in D_{ps}'$ is a plane strain solution to the boundary-initial-value problem stated in section 2 then condition $(V)_p$ is equivalent to

$$c(2\sigma - \int_{\Gamma} \bar{e}_1 \cdot \bar{\Psi}_p \bar{n} ds(x)) = 0 \quad (4.19)$$

where

$$\bar{\Psi}_p \equiv w \bar{I}_p - \nabla \tilde{u}^T \bar{I}_p \quad (4.20)$$

If $u \in D_{as}$ is an anti-plane shear solution to the boundary-initial-value problem then condition $(V)_a$ is equivalent to

$$c(2\sigma - \mu \int_{\Gamma} \left(\frac{1}{2} |\nabla u|^2 \bar{e}_1 \cdot \bar{n} - (\bar{e}_1 \cdot \nabla u)(\bar{n} \cdot \nabla u) \right) ds(x)) \quad (4.21)$$

where, in both cases, Γ is any regular curve in $\mathcal{D}$ which begins and ends on the crack and surrounds the crack-tip (cf. Figure 3.2).

Proof: Since

$$\dot{\tilde{u}} = \tilde{u}' - c \frac{\partial \tilde{u}}{\partial x_1}, \quad \dot{\tilde{I}} = \tilde{I}' - c \frac{\partial \tilde{I}}{\partial x_1}$$

Sanders' I-integral may be written

$$\dot{I} = \frac{c}{2} \int_{\Gamma} \left(\tilde{u} \cdot \frac{\partial \tilde{I}}{\partial x_1} \bar{n} - \frac{\partial \tilde{u}}{\partial x_1} \cdot \tilde{I} \bar{n} \right) ds(x) + Q$$

where

$$Q \equiv \int_{\Gamma} (\tilde{u}' \cdot \tilde{I} \bar{n} - \tilde{u} \cdot \tilde{I}' \bar{n}) ds(x)$$

We know that I is path-independent. However, since

$$(\tilde{u}') \cdot \tilde{\Gamma}' - (\tilde{\Gamma}')' \tilde{u}) = 0, \quad x \in \mathcal{D} \setminus C_t$$

$$(\tilde{u}') \cdot \tilde{\Gamma}' - (\tilde{\Gamma}')' \tilde{u}) \cdot \bar{e}_2 = 0, \quad q \in [0, \ell(t)]$$

Q is also path-independent. Thus we have, for $0 < R \leq R_0$ ($R_0 > 0$ suitably chosen)

$$Q = \int_{O_R} (\tilde{u}' \cdot \tilde{\Gamma} \bar{n} - \tilde{u} \cdot \tilde{\Gamma}' \bar{n}) ds(x) = \int_{-\pi}^{\pi} (\tilde{u}'(R, \theta, t) \cdot \tilde{\Gamma}(R, \theta, t) \bar{e}_r - \tilde{u}(R, \theta, t) \cdot \tilde{\Gamma}'(R, \theta, t) \bar{e}_r) R d\theta \quad (4.22)$$

where O_R is the circle defined in (4.11). But Q is independent of R so, by integrating both sides of equation (4.22) with respect to R we get

$$Q R_0 = \int_0^{R_0} \left(\int_{-\pi}^{\pi} (\tilde{u}' \cdot \tilde{\Gamma} \bar{e}_r - \tilde{u} \cdot \tilde{\Gamma}' \bar{e}_r) R d\theta \right) dR$$

From our previous discussions we know that $\tilde{u}$, $\tilde{u}'$, $\tilde{\Gamma}$ and $\tilde{\Gamma}' \in L^2(\mathcal{D})$ and thus, from Fubini's Theorem¹, we get

$$Q R_0 = \int_{\Delta(R_0, t)} (\tilde{u}' \cdot \tilde{\Gamma} \bar{e}_r - \tilde{u} \cdot \tilde{\Gamma}' \bar{e}_r) da(x)$$

From Cauchy-Schwartz inequality we obtain the estimate

$$|Q| R_0 \leq \left(\int_{\Delta(R_0, t)} |\tilde{u}'|^2 da(x) \right)^{\frac{1}{2}} \left(\int_{\Delta(R_0, t)} |\tilde{\Gamma}|^2 da(x) \right)^{\frac{1}{2}} + \left(\int_{\Delta(R_0, t)} |\tilde{u}|^2 da(x) \right)^{\frac{1}{2}} \left(\int_{\Delta(R_0, t)} |\tilde{\Gamma}'|^2 da(x) \right)^{\frac{1}{2}}$$

But, from Corollary 1 and (3.1), we get

$$\int_{\Delta(R_0, t)} |\tilde{u}'|^2 da(x) \leq C_1 R_0^2, \quad \int_{\Delta(R_0, t)} |\tilde{u}|^2 da(x) \leq C_2 R_0^2$$

where C_1, C_2 are positive constants (possibly depending on t). Consequently

$$|Q| \leq (C_1 \int_{\Delta(R_0, t)} |\tilde{\Gamma}|^2 da(x))^{\frac{1}{2}} + (C_2 \int_{\Delta(R_0, t)} |\tilde{\Gamma}'|^2 da(x))^{\frac{1}{2}}$$

¹ See Schwartz [8], Theorem 28 on p. 34.

But since $R_0 > 0$ may be taken arbitrarily small we must have

$$Q = 0$$

Thus

$$I = \frac{c}{2} \int_{\Gamma} (\bar{u} \cdot \frac{\partial \bar{T}}{\partial x_1} \bar{n} - \frac{\partial \bar{u}}{\partial x_1} \cdot \bar{T} \bar{n}) ds(x) = \frac{c}{2} \int_{O_R} (\bar{u} \cdot \frac{\partial \bar{T}}{\partial x_1} \bar{n} - \frac{\partial \bar{u}}{\partial x_1} \cdot \bar{T} \bar{n}) ds(x) =$$

$$\frac{c}{2} \int_{-\pi}^{\pi} (\bar{u} \cdot \frac{\partial \bar{T}}{\partial x_1} \bar{e}_r - \frac{\partial \bar{u}}{\partial x_1} \cdot \bar{T} \bar{e}_r) R d\theta$$

(arguments (R, θ, t) , in the last integral, are omitted). From the relations

$$\frac{\partial \bar{u}}{\partial x_1} = \frac{\partial \bar{u}}{\partial r} \cos \theta - \frac{\partial \bar{u}}{\partial \theta} \frac{\sin \theta}{r}, \quad \frac{\partial \bar{T}}{\partial x_1} = \frac{\partial \bar{T}}{\partial r} \cos \theta - \frac{\partial \bar{T}}{\partial \theta} \frac{\sin \theta}{r}$$

$$\operatorname{div} \bar{T} = \frac{\partial \bar{T}}{\partial r} \bar{e}_r + \frac{1}{r} \frac{\partial \bar{T}}{\partial \theta} \bar{e}_\theta = \bar{0}, \quad \bar{T} \cdot \nabla \bar{u} = \frac{\partial \bar{u}}{\partial r} \cdot \bar{T} \bar{e}_r + \frac{1}{r} \frac{\partial \bar{u}}{\partial \theta} \cdot \bar{T} \bar{e}_\theta$$

(where $\bar{e}_\theta = \bar{e}_1(-\sin \theta) + \bar{e}_2 \cos \theta$) we get

$$\bar{u} \cdot \frac{\partial \bar{T}}{\partial x_1} \bar{e}_r = -\frac{1}{r} \frac{\partial}{\partial \theta} (\bar{u} \cdot \bar{T} \bar{e}_2) + \bar{T} \cdot \nabla \bar{u} \cos \theta - \frac{\partial \bar{T}}{\partial x_1} \cdot \bar{T} \bar{e}_r$$

and since $\bar{T}(R, \pm\pi, t) \bar{e}_2 = \bar{0}$ we have

$$I = c \int_{-\pi}^{\pi} (\frac{1}{2} \bar{T} \cdot \nabla \bar{u} \cos \theta - \frac{\partial \bar{u}}{\partial x_1} \cdot \bar{T} \bar{e}_r) R d\theta = c \int_{O_R} (\frac{1}{2} \bar{T} \cdot \nabla \bar{u} \bar{e}_1 \cdot \bar{n} - \frac{\partial \bar{u}}{\partial x_1} \cdot \bar{T} \bar{n}) ds(x)$$

$$= c \int_{O_R} \bar{e}_1 \cdot \bar{T} \bar{n} ds(x)$$

where $\bar{\Psi} = w \bar{I} - \nabla \bar{u}^T \bar{T}$. In plane strain we have $\bar{\Psi} = \bar{\Psi}_p + \bar{e}_3 \otimes \bar{e}_3 (w - \lambda \operatorname{tr} \bar{\Psi})$ and from Eshelby's relation

$$\operatorname{div} \bar{\Psi}_p = \bar{0}, \quad x \in \mathcal{D} \setminus C_t$$

and the fact that

$$\bar{e}_1 \cdot (\bar{\Psi}_p)^\pm \bar{e}_2 = 0, \quad q \in [0, \ell(t)[$$

it follows that the integral

$$\int_{\Gamma} \bar{e}_1 \cdot \underline{\Psi}_p \bar{n} ds(x)$$

is path-independent. Consequently

$$I = c \int_{\Gamma} \bar{e}_1 \cdot \underline{\Psi}_p \bar{n} ds(x) \quad (4.23)$$

and (4.19) now follows from (4.23) and (3.5). In anti-plane shear we have $\underline{\Psi} = w \underline{1} - \nabla \bar{u}^T \underline{1} = (\mu/2) |\nabla u|^2 \underline{1} - \mu \nabla u \otimes \nabla u$. $\square$

Remark: The integral

$$J = J(t) \equiv \int_{\Gamma} \bar{e}_1 \cdot \underline{\Psi} \bar{n} ds(x)$$

is the celebrated path-independent J-integral introduced into fracture mechanics by Rice (see Rice [9]). The tensor $\underline{\Psi}$ may be called the quasi-static energy-momentum tensor (see Eshelby [10]).

With presumptions as in Theorem 1 we have the following simple Corollary.

Corollary 2 a) If $c(t) = 0$ (the crack is resting at time t) then condition $(V)_p$ is satisfied automatically.

b) If $c(t) \neq 0$ then condition $(V)_p$ is equivalent to the condition

$$2\sigma = J \quad (4.24)$$

Proof: This is an immediate consequence of formula (4.19). $\square$

5. CONCLUDING REMARKS

If we look for an anti-plane shear solution in the set D_{as} corresponding to an advancing crack ($c(t) > 0$, $t \in [0, T]$) then, according to Corollary 2, the boundary-initial-value problem stated in section 2 is equivalent to the problem:

Find $u \in D_{as}$, with $c(t) > 0$ for $t \in [0, T]$, so that

$$\Delta u = 0, \quad x \in \mathcal{D} \setminus C_t \quad (I)_a$$

$$\left(\frac{\partial u}{\partial x_2}\right)^{\pm} = 0, \quad q \in [0, \ell(t)[\quad (II)_a$$

$$2\sigma = J \quad (4.24)$$

$$\mu \bar{n} \cdot \nabla u = t_0, \quad x \in \partial \mathcal{D} \quad (VI)_a$$

$$\ell(0) = \ell_0$$

$$u(x, 0) = 0 \quad (VII)_a$$

$$\dot{u}(x, 0) = 0 \quad x \in \mathcal{D} \setminus C_{t=0}$$

where $\ell_0 > 0$ and t_0 satisfies (2.16).

Due to our quasi-static approach this crack-problem exhibits several anomalies. First of all the time-variable plays a rather subordinate role. The initial-value conditions, in $(VII)_a$, are the only equations involving time explicitly. Moreover, the crack-tip speed c is nowhere explicitly present in the equations. The motion of the crack is only involved in the determination of the regions $\mathcal{D} \setminus C_t$ and $[0, \ell(t)[$. These anomalies are reflected in the fact that if $u = u(x, t)$ is a solution then $u_1(x, t) = u(x, t) + f(t)$ is a solution for every C^2 -function $f: [0, T] \rightarrow \mathbb{R}$ satisfying $f(0) = 0$, $\dot{f}(0) = 0$.

One way of utilizing this problem in engineering practice is to forget about the time-variable altogether and solve the static problem:

Find $u = u(x)$ so that

$$\Delta u = 0, \quad x \in \mathcal{D} \setminus C_0$$

$$\left(\frac{\partial u}{\partial x_2}\right)^{\pm} = 0, \quad q \in [0, \ell_0[$$

$$\mu \bar{n} \cdot \nabla u = t_0, \quad x \in \partial \mathcal{D}$$

This corresponds to an equilibrium situation prior to fracture. The condition

$$2\sigma = J$$

is now interpreted as a criterion for the initiation of crack-growth. Finally we point out that the analysis performed in this paper may be extended to the case of time-dependent, mixed boundary conditions. One is easily convinced that Propositions 2 and 3 and Theorem 1 are still valid.

APPENDIX

With $\mathcal{D}$ and $\mathcal{C}$ defined as in the main text we have (the time-parameter is fixed in this section)

Theorem A.1 Let

$$\underline{A} : \mathcal{D} \setminus \mathcal{C} \rightarrow \text{End}(\mathbb{R}^3), \quad \bar{v} : \mathcal{D} \setminus \mathcal{C} \rightarrow \mathbb{R}^3$$

be fields satisfying:

$$(\alpha) \quad \underline{A} \text{ and } \bar{v} \in C^1(\mathcal{D} \setminus \mathcal{C})$$

$$(\beta) \quad \text{Limits } (\underline{A})^+, (\underline{A})^-, (\bar{v})^+, (\bar{v})^-, (\nabla \bar{v})^+ \text{ and } (\nabla \bar{v})^-, \text{ in the sense of (2.9), exist for } q \in [0, \ell[.$$

$$(\gamma) \quad (\underline{A})^+ \bar{e}_2 = (\underline{A})^- \bar{e}_2 = \bar{0}, \quad q \in [0, \ell[$$

$$(\delta) \quad \text{div } \underline{A} = \bar{0}, \quad x \in \mathcal{D} \setminus \mathcal{C}$$

$$(\epsilon) \quad \frac{1}{2\pi R} \int_{O_R} |\bar{v}|^2 ds(x), \quad R > 0$$

remains bounded as $R \rightarrow 0$, where

$$O_R = \{x \in \mathbb{R}^2 \mid (x_1 - \ell)^2 + x_2^2 = R^2\}$$

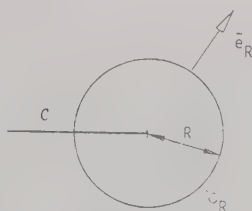


Figure A.1

$$(\zeta) \quad \underline{A} \cdot \nabla \bar{v} \geq C |\underline{A}|^2, \quad x \in \mathcal{D} \setminus \mathcal{C}$$

for some constant $C > 0$.

$$\text{Then } \underline{A} \cdot \nabla \bar{v} \in L^1(\mathcal{D}) \text{ and } \int_{\mathcal{D}} \underline{A} \cdot \nabla \bar{v} da(x) = \int_{\partial \mathcal{D}} \bar{v} \cdot \underline{A} \bar{n} ds(x) \quad (\text{A.1})$$

Proof: Put, for $R > 0$,

$$D_R \equiv \{x \in \mathbb{R}^2 \mid (x_1 - \ell)^2 + x_2^2 \leq R^2\}, \quad \mathcal{D}_R \equiv \mathcal{D} \setminus (D_R \cup C)$$

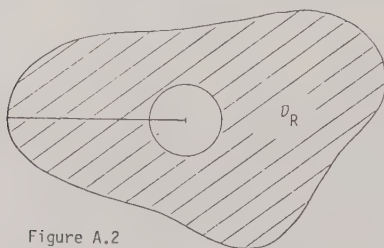


Figure A.2

Following Knowles & Pucik [2] we introduce the function

$$]0, R_0] \ni R \mapsto f(R) \equiv \int_{\mathcal{D}_R} \underline{A} \cdot \nabla \bar{v} \, da(x) - \int_{\partial \mathcal{D}} \bar{v} \cdot \underline{A} \bar{n} \, ds(x) \quad (A.2)$$

(We take R_0 small enough to assure that $D_{R_0} \subset \bar{\mathcal{D}}$). It is a simple matter to show that f is a C^1 -function with the derivative

$$\frac{df}{dR}(R) = - \int_{O_R} \underline{A} \cdot \nabla \bar{v} \, ds(x) \quad (A.3)$$

and from (z) it follows that

$$\frac{df}{dR}(R) \leq 0, \quad R \in]0, R_0]$$

A simple calculation using (δ) gives

$$\underline{A} \cdot \nabla \bar{v} = \operatorname{div}(\underline{A}^T \bar{v}) \quad (A.4)$$

and by applying the divergence theorem in $\mathbb{R}^2$ to the region $\mathcal{D}_R$ and the field $\underline{A}^T \bar{v}$ it follows, from (A.2), (A.4) and (γ), that

$$f(R) = - \int_{O_R} \bar{v} \cdot \underline{A} \bar{e}_R \, ds(x)$$

where $\bar{e}_R$ is the outward unit normal to the circle O_R . From Cauchy-Schwartz inequality and (z) it follows that

$$|f(R)| \leq \int_{O_R} |\bar{v}| |\underline{A}| \, ds(x) \leq \left(\int_{O_R} |\bar{v}|^2 \, ds(x) \right)^{\frac{1}{2}} \left(\int_{O_R} |\underline{A}|^2 \, ds(x) \right)^{\frac{1}{2}} \leq \\ \left(\int_{O_R} |\bar{v}|^2 \, ds(x) \right)^{\frac{1}{2}} \left(\frac{1}{C} \int_{O_R} \underline{A} \cdot \nabla \bar{v} \, ds(x) \right)^{\frac{1}{2}} \quad (A.5)$$

According to (e) there is a constant $B > 0$ satisfying

$$\int_{O_R} |\bar{v}|^2 \, ds(x) \leq BR \quad \text{for all } R \in]0, R_0] \quad (A.6)$$

Finally, from (A.3), (A.5) and (A.6) we obtain the differential inequality

$$|f(R)| \leq \left(\frac{B}{C} \left(-R \frac{df}{dR}(R) \right) \right)^{\frac{1}{2}}$$

and since $\underline{A} \cdot \nabla \bar{v} \geq 0$ the Theorem follows from

Lemma A.1 (Knowles & Pucik) Let $f:]0, R_0] \rightarrow \mathbb{R}$ be a C^1 -function satisfying

$$(n) \quad \frac{df}{dR}(R) \leq 0, \quad R \in]0, R_0]$$

$$(0) \quad |f(R)| \leq C \left(-R \frac{df}{dR}(R) \right)^{\frac{1}{2}}, \quad R \in]0, R_0]$$

for some positive constant C .

Then $\lim_{R \rightarrow 0} f(R) = 0$.

Proof of Lemma A.1: See Knowles & Pucik [2]. $\square$

Remark: With $\Omega(R_0)$ as in the main text (see (4.4)) we have the following simple reformulation of Theorem A.1: Let

$$\tilde{\underline{A}}: \Omega(R_0) \rightarrow \text{End}(\mathbb{R}^3), \quad \tilde{\bar{v}}: \Omega(R_0) \rightarrow \mathbb{R}^3$$

be fields satisfying:

$$(\alpha) \quad \tilde{\underline{A}} \text{ and } \tilde{\bar{v}} \in C^1(\Omega(R_0))$$

(β) Limits $(\tilde{\underline{A}})^+$, $(\tilde{\underline{A}})^-$, $(\tilde{\bar{v}})^+$, $(\tilde{\bar{v}})^-$, $(\nabla \tilde{\bar{v}})^+$ and $(\nabla \tilde{\bar{v}})^-$, in the sense of (4.8), exist for $\rho \in]0, R_0]$.

$$(\gamma) \quad (\tilde{A})^+ \bar{e}_2 = (\tilde{A})^- \bar{e}_2 = \bar{0}, \quad \rho \in]0, R_0]$$

$$(\delta) \quad \operatorname{div} \tilde{A} = \bar{0}, \quad \bar{r} \in \Omega(R_0)$$

$$(\varepsilon) \quad \int_{-\pi}^{\pi} |\tilde{v}(R, \theta)|^2 d\theta, \quad R > 0$$

remains bounded as $R \rightarrow 0$.

$$(\zeta) \quad \tilde{A} \cdot \nabla \tilde{v} \geq C |\tilde{A}|^2, \quad \bar{r} \in \Omega(R_0)$$

for some constant $C > 0$.

Then $\tilde{A} \cdot \nabla \tilde{v} \in L^1(\Omega(R_0))$ and for $0 < R \leq R_0$ we have

$$\int_{-\pi}^{\pi} \int_0^R \tilde{A}(r, \theta) \cdot \nabla \tilde{v}(r, \theta) r dr d\theta = R \int_{-\pi}^{\pi} \tilde{v}(R, \theta) \cdot \tilde{A}(R, \theta) \bar{e}_r d\theta \quad (A.7)$$

where $\bar{e}_r = \bar{e}_1 \cos \theta + \bar{e}_2 \sin \theta$.

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PAPER IV

ON THE UNIQUENESS FOR PLANE CRACK PROBLEMS
IN LINEAR ELASTOSTATICS

by

Per Lidström

ABSTRACT

In this paper we consider the traction boundary value problem in linearized elasto-statics for a two-dimensional body containing an internal crack. The material may be non-homogeneous and anisotropic. The model includes the possibility of non-zero surface tensions on the crack-surfaces.

The paper contains a proof of the uniqueness of the solution to the traction boundary value problem within a class of displacement fields that give rise to moderate singularities of the stress field.

1. INTRODUCTION

The classical traction problem of elastostatics is to determine the equilibrium displacement field produced, in a body of known reference shape and known elastic constitution, by the action of known body forces and surface tractions.

In linear elastostatics and for a body with regular boundary, uniqueness of the traction problem, within the class of smooth displacement fields, was demonstrated by Kirchhoff [1].¹

In this paper we state a traction problem in linear elastostatics for a two-dimensional body containing a straight line crack. We prove uniqueness within a class of solutions that may be singular (unbounded) at the crack-tips. The model presented here is more general than the one usually encountered in the literature (cf. Sneddon & Lowengrub [3]) in that we allow for non-zero surface tensions on the crack-surfaces. Our approach is to some extent based on Lidström [4].

Notation: The analysis takes place in a two-dimensional Euclidean space $\mathbb{R}^2$. Points in $\mathbb{R}^2$ are denoted: $x, y, \dots$, vectors: $\vec{a}, \vec{b}, \dots$ and tensors ("linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ "): $\underline{A}, \underline{B}, \dots$; $\underline{A}^T$ is the transpose of $\underline{A}$. In standard indicial notation and rectangular Cartesian coordinates we have:

$$\vec{a} \cdot \vec{b} = a_i b_i, \quad \underline{A} \cdot \underline{B} = A_{ij} B_{ij}, \quad (i, j = 1, 2)^2$$

$\text{div } \underline{T}$ is a vector with components $\partial T_{ij} / \partial x_j$

$\nabla \vec{u}$ is a tensor with components $\partial u_i / \partial x_j$

$$\text{End}(\mathbb{R}^2) = \{\text{tensors in } \mathbb{R}^2\}$$

$$\text{Sym}(\mathbb{R}^2) = \{\underline{A} \in \text{End}(\mathbb{R}^2) \mid \underline{A}^T = \underline{A}\}$$

$$C^m(B) = \{m\text{-continuously differentiable functions on } B\}$$

da denotes (Lebesgue) area measure and ds arc-length measure.

$$L^1(B) = \{\text{integrable functions on } B\}$$

¹ see also [2]

² the usual summation convention is employed.

2. BASIC EQUATIONS

A two-dimensional body is identified with a reference configuration B , a bounded regular region in $\mathbb{R}^2$. Let $\bar{n}$ denote the outward unit normal on ∂B . The crack is represented by a straight-lined segment C lying in B . Choose a rectangular Cartesian coordinate system $(\bar{e}_1, \bar{e}_2)$ with origin O , $x = (x_1, x_2)$ in $\mathbb{R}^2$ so that

$$C = \{x \in \mathbb{R}^2 \mid -c \leq x_1 \leq c, x_2 = 0\} \quad (2.1)$$

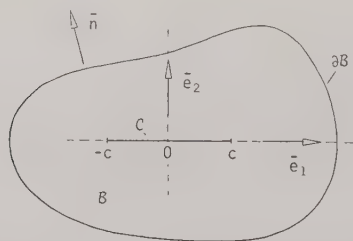


Figure 2.1

Let

$$\bar{u} : B \setminus C \rightarrow \mathbb{R}^2 \quad (2.2)$$

denote the displacement of the body. The displacement of the crack is given by

$$\bar{u}^+ : [-c, c] \rightarrow \mathbb{R}^2, \quad \bar{u}^- : [-c, c] \rightarrow \mathbb{R}^2 \quad (2.3)$$

where

$$\begin{aligned} \bar{u}^+(q) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 > 0}} \bar{u}(x), & \bar{u}^-(q) &\equiv \lim_{\substack{x \rightarrow (q, 0) \\ x_2 < 0}} \bar{u}(x), & q &\in]-c, c[\end{aligned} \quad (2.4)$$

$$\begin{aligned} \bar{u}^+(c) = \bar{u}^-(c) &\equiv \lim_{\substack{x \rightarrow (c, 0) \\ x \in B \setminus C}} \bar{u}(x), & \bar{u}^+(-c) = \bar{u}^-(-c) &\equiv \lim_{\substack{x \rightarrow (-c, 0) \\ x \in B \setminus C}} \bar{u}(x) \end{aligned} \quad (2.5)$$

Denote by D the set of displacement fields $\bar{u} \in C^2(B \setminus C)$ satisfying

(i) limits $(\nabla \bar{u})^+$, $(\nabla \bar{u})^-$, in the sense of (2.4), exist for $q \in]-c, c[$.

(ii) $\bar{u}^+$, $\bar{u}^- \in C^2([-c, c])$

(iii) the "crack-surfaces" ¹ in the deformed configuration

$$C^+ \equiv \{x \in \mathbb{R}^2 \mid x = (q, 0) + \bar{u}^+(q), \quad q \in [-c, c]\}$$

$$C^- \equiv \{x \in \mathbb{R}^2 \mid x = (q, 0) + \bar{u}^-(q), \quad q \in [-c, c]\}$$

are nowhere in contact (except, of course, at the crack-tips, i.e. at the points $(\pm c, 0)$).

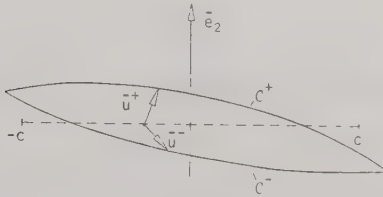


Figure 2.2

Remark: Note that $\nabla \bar{u}$, $\nabla^2 \bar{u}$, ... may be singular (unbounded) at the crack-tips.

Let

$$\underline{T} : B \setminus C \rightarrow \text{End}(\mathbb{R}^2) \quad (2.6)$$

denote the "bulk" (Piola-Kirchhoff) stress tensor and let

$$S^+ : [-c, c] \rightarrow \mathbb{R}, \quad S^- : [-c, c] \rightarrow \mathbb{R} \quad (2.7)$$

denote the "crack-surface" tensions.

Remark: The (Piola-Kirchhoff) "crack-surface" stress tensor may be written

$$\underline{S}^\pm = S^\pm \bar{e}^\pm \otimes \bar{e}_1$$

¹ "crack-curves" would perhaps be more adequate in our two-dimensional context.

where $\bar{e}^\pm$ is a unit tangent vector field (an orientation) on $C^\pm$.¹

In the linearized theory the elastic material under consideration is defined by a smooth elasticity field on B :

$$\Phi: B \rightarrow L(\text{End}(\mathbb{R}^2), \text{Sym}(\mathbb{R}^2)) \quad (2.8)$$

(cf. Gurtin [5], section 20), and smooth elasticity fields on the "crack-surfaces":

$$k^+: [-c, c] \rightarrow \mathbb{R}, \quad k^-: [-c, c] \rightarrow \mathbb{R} \quad (2.9)$$

We assume that

(iv) Φ is positive definite, i.e.

$$\underline{A} \cdot \Phi(x)[\underline{A}] > 0$$

$x \in B$ and every non-zero $\underline{A} \in \text{Sym}(\mathbb{R}^2)$.

(v) $k^+(q) \geq 0, \quad k^-(q) \geq 0, \quad q \in [-c, c]$

We may now formulate the traction problem for B in linear elastostatics: Find $\bar{u} \in D$ so that

$$\text{div } \underline{I} + \bar{b} = \bar{0}, \quad x \in B \setminus C \quad (2.10)$$

$$\underline{I} = \Phi[\underline{E}] \quad (2.11)$$

$$\underline{E} = \frac{1}{2} (\nabla \bar{u} + \nabla \bar{u}^T) \quad (2.12)$$

$$\frac{dS^+}{dq} \bar{e}_1 + (\underline{I})^+ \bar{e}_2 = \bar{0} \quad q \in]-c, c[\quad (2.13)$$

$$\frac{dS^-}{dq} \bar{e}_1 - (\underline{I})^- \bar{e}_2 = \bar{0} \quad (2.14)$$

$$S^+ = k^+ \frac{du_1^+}{dq}, \quad S^- = k^- \frac{du_1^-}{dq} \quad (2.15)$$

$$u_1^+ \equiv \bar{e}_1 \cdot \bar{u}^+, \quad u_1^- \equiv \bar{e}_1 \cdot \bar{u}^- \quad (2.16)$$

¹ $\otimes$ denotes the tensor-product: $(\bar{a} \otimes \bar{b}) \bar{u} = \bar{a} \bar{b} \cdot \bar{u}$

$$(S^+(c) + S^-(c)) \bar{e}_1 = \lim_{R \rightarrow 0} \int_{O_R^1} \underline{I} \bar{e}_R^1 ds \quad (2.17)$$

$$-(S^+(-c) + S^-(-c)) \bar{e}_1 = \lim_{R \rightarrow 0} \int_{O_R^1} \underline{I} \bar{e}_R^2 ds \quad (2.18)$$

$$\bar{O} = \lim_{R \rightarrow 0} R \int_{O_R^1} \bar{e}_R^1 \times \underline{I} \bar{e}_R^1 ds \quad (2.19)^1$$

$$\bar{O} = \lim_{R \rightarrow 0} R \int_{O_R^1} \bar{e}_R^2 \times \underline{I} \bar{e}_R^2 ds \quad (2.20)$$

$$\underline{I} \bar{n} = \bar{t}, \quad x \in \partial B \quad (2.21)$$

where

(vi) the body force

$$\bar{b}: B \rightarrow \mathbb{R}^2 \quad (2.22)$$

and the boundary traction

$$\bar{t}: \partial B \rightarrow \mathbb{R}^2 \quad (2.23)$$

are prescribed continuous fields satisfying (x_0 a fix point in $\mathbb{R}^2$)

$$\int_{\partial B} \bar{t} ds + \int_B \bar{b} da = \bar{O} \quad (2.24)$$

$$\int_{\partial B} (x - x_0) \times \bar{t} ds + \int_B (x - x_0) \times \bar{b} da = \bar{O}$$

and

$$\begin{aligned} O_R^1 &= \{x \in \mathbb{R}^2 \mid (x_1 - c)^2 + x_2^2 = R^2\} \\ O_R^2 &= \{x \in \mathbb{R}^2 \mid (x_1 + c)^2 + x_2^2 = R^2\} \end{aligned} \quad (2.25)$$

$\bar{e}_R^1$ and $\bar{e}_R^2$ are outward unit normals on the circles O_R^1 and O_R^2 respectively. See Figure 2.3!

¹ $\times$ denotes the vector product in $\mathbb{R}^3$.

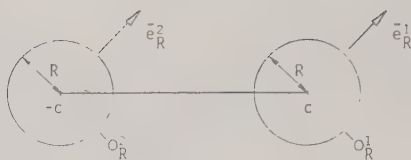


Figure 2.3

Remark 1: It may be shown that equations (2.10) - (2.21) are equivalent to vanishing force and moment on every part (subbody) of B in the linear theory. See Lidström [4].

Remark 2: Note that (2.24) is a necessary condition for the traction problem to have a solution.

Remark 3: Equations (2.13), (2.14) are valid under the assumption that the "crack-surfaces" C^+ and C^- are nowhere in contact (cf. condition (iii)). If this restriction is removed, so that portions of the "crack-surfaces" may be in contact, then we have to introduce fields

$$\bar{f}^+ :]-c, c[\rightarrow \mathbb{R}^2, \quad \bar{f}^- :]-c, c[\rightarrow \mathbb{R}^2 \quad (2.26)$$

representing contact forces between the "crack-surfaces". Equations (2.13), (2.14) then change for

$$\begin{aligned} \frac{dS^+}{dq} \bar{e}_1 + (I)^+ \bar{e}_2 + \bar{f}^+ &= \bar{0} \\ \frac{dS^-}{dq} \bar{e}_1 - (I)^- \bar{e}_2 + \bar{f}^- &= \bar{0} \end{aligned} \quad q \in]-c, c[\quad (2.27)$$

3. THEOREM OF WORK AND ENERGY

Let $\bar{u}$ be a solution to the traction problem stated in section 2, satisfying

$$(vii) \quad \int_{O_R^1} |\bar{I} \bar{e}_R^1| \, ds \quad \text{and} \quad \int_{O_R^2} |\bar{I} \bar{e}_R^2| \, ds$$

remain bounded as $R \rightarrow 0$.

Then

$$\int_{\bar{B}} \bar{E} \cdot \Phi[\bar{E}] \, da + \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) \, dq = \int_{\partial \bar{B}} \bar{t} \cdot \bar{u} \, ds + \int_{\bar{B}} \bar{b} \cdot \bar{u} \, da$$

Proof: Put, for $R > 0$

(3.1)

$$D_R^1 = \{x \in \mathbb{R}^2 \mid (x_1 - c)^2 + x_2^2 \leq R^2\}$$

$$D_R^2 = \{x \in \mathbb{R}^2 \mid (x_1 + c)^2 + x_2^2 \leq R^2\}$$

and

$$B_R = B \setminus (C \cup D_R^1 \cup D_R^2)$$

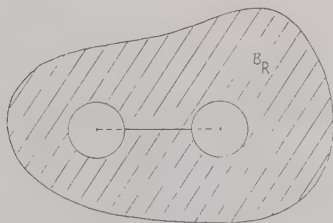


Figure 3.1

From (2.10) - (2.12) we infer that

$$\bar{E} \cdot \Phi[\bar{E}] = \operatorname{div} (\bar{I} \bar{u}) + \bar{b} \cdot \bar{u}, \quad x \in B \setminus C \quad (3.2)$$

and the divergence theorem gives (we assume that $R > 0$ is sufficiently small so that we have the situation illustrated in Figure 3.1 above)

$$\begin{aligned}
\int_{\bar{B}_R} \underline{E} \cdot \Phi[\underline{E}] \, da &= \int_{\partial \bar{B}} \bar{n} \cdot \underline{I} \bar{u} \, ds - \int_{O_R^1} \bar{e}_R^1 \cdot \underline{I} \bar{u} \, ds - \int_{O_R^2} \bar{e}_R^2 \cdot \underline{I} \bar{u} \, ds - \\
&\int_{-c+R}^{c-R} \bar{e}_2 \cdot ((\underline{I} \bar{u})^+ - (\underline{I} \bar{u})^-) \, dq + \int_{\bar{B}_R} \bar{b} \cdot \bar{u} \, da \quad (3.3)
\end{aligned}$$

From (2.13) - (2.16) we get

$$\begin{aligned}
\bar{e}_2 \cdot ((\underline{I} \bar{u})^+ - (\underline{I} \bar{u})^-) &= \bar{u}^+ \cdot (\underline{I})^+ \bar{e}_2 - \bar{u}^- \cdot (\underline{I})^- \bar{e}_2 = -u_1^+ \frac{dS^+}{dq} - u_1^- \frac{dS^-}{dq} = \\
&k^+ \left(\frac{du_1^+}{dq} \right)^2 + k^- \left(\frac{du_1^-}{dq} \right)^2 - \frac{d}{dq} (u_1^+ S^+ + u_1^- S^-), \quad q \in]-c, c[
\end{aligned}$$

and then (3.3) may be written

$$\begin{aligned}
\int_{\bar{B}_R} \underline{E} \cdot \Phi[\underline{E}] \, da &= \int_{\partial \bar{B}} \bar{n} \cdot \underline{I} \bar{u} \, ds - \int_{O_R^1} \bar{e}_R^1 \cdot \underline{I} \bar{u} \, ds - \int_{O_R^2} \bar{e}_R^2 \cdot \underline{I} \bar{u} \, ds - \\
&\int_{-c+R}^{c-R} (k^+ \left(\frac{du_1^+}{dq} \right)^2 + k^- \left(\frac{du_1^-}{dq} \right)^2) \, dq + u_1^+(c-R) S^+(c-R) + u_1^-(c-R) S^-(c-R) - \\
&u_1^+(-c+R) S^+(-c+R) - u_1^-(-c+R) S^-(-c+R) + \int_{\bar{B}_R} \bar{b} \cdot \bar{u} \, da = \int_{\partial \bar{B}} \bar{n} \cdot \underline{I} \bar{u} \, ds - \\
&\int_{-c+R}^{c-R} (k^+ \left(\frac{du_1^+}{dq} \right)^2 + k^- \left(\frac{du_1^-}{dq} \right)^2) \, dq + (u_1^+(c-R) - u_1^+(c)) S^+(c-R) + (u_1^-(c-R) - \\
&u_1^-(c)) S^-(c-R) + ((S^+(c-R) + S^-(c-R)) \bar{e}_1 - \int_{O_R^1} \underline{I} \bar{e}_R^1 \, ds) \cdot \bar{u}^+(c) + \\
&\int_{O_R^1} \bar{e}_R^1 \cdot \underline{I} (\bar{u}^+(c) - \bar{u}) \, ds + (u_1^+(-c) - u_1^+(-c+R)) S^+(-c+R) + (u_1^-(-c) - \\
&u_1^-(-c+R)) S^-(-c+R) + (- (S^+(-c+R) + S^-(-c+R))) \bar{e}_1 - \int_{O_R^2} \underline{I} \bar{e}_R^2 \, ds) \cdot \bar{u}^+(-c) + \\
&\int_{O_R^2} \bar{e}_R^2 \cdot \underline{I} (\bar{u}^+(-c) - \bar{u}) \, ds + \int_{\bar{B}_R} \bar{b} \cdot \bar{u} \, ds \quad (3.4)
\end{aligned}$$

(Note that $\bar{u}^+(c) = \bar{u}^-(c)$ and $\bar{u}^+(-c) = \bar{u}^-(-c)$)

From (vii) it follows that

$$\lim_{R \rightarrow 0} \left| \int_{O_R^1} \bar{e}_R^1 \cdot \underline{I} (\bar{u}^+(c) - \bar{u}) ds \right| \leq \lim_{R \rightarrow 0} \sup_{x \in O_R^1} |\bar{u}^+(c) - \bar{u}(x)| \int_{O_R^1} |\underline{I} \bar{e}_R^1| ds = 0$$

$$\lim_{R \rightarrow 0} \left| \int_{O_R^2} \bar{e}_R^2 \cdot \underline{I} (\bar{u}^+(-c) - \bar{u}) ds \right| \leq \lim_{R \rightarrow 0} \sup_{x \in O_R^2} |\bar{u}^+(-c) - \bar{u}(x)| \int_{O_R^2} |\underline{I} \bar{e}_R^2| ds = 0$$

and thus by taking limits in (3.4) and by invoking (2.17) and (2.18) we get

$$\lim_{R \rightarrow 0} \int_B \underline{E} \cdot \Phi[\underline{E}] da = \int_{\partial B} \bar{u} \cdot \underline{I} \bar{n} ds - \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) dq + \int_B \bar{b} \cdot \bar{u} da$$

and since $\underline{E} \cdot \Phi[\underline{E}] \geq 0$ we may conclude that $\underline{E} \cdot \Phi[\underline{E}] \in L^1(B)$ and that

$$\int_B \underline{E} \cdot \Phi[\underline{E}] da = \int_{\partial B} \bar{u} \cdot \underline{I} \bar{n} ds - \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) dq + \int_B \bar{b} \cdot \bar{u} da$$

and from this and (2.21) we obtain formula (3.1). $\square$

Remark 1: We may call

$$\frac{1}{2} \int_B \underline{E} \cdot \Phi[\underline{E}] da + \frac{1}{2} \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) dq$$

the strain energy of the body.

Remark 2: Note that the conditions (2.19) and (2.20), in the traction problem, are satisfied automatically if we assume that the stress tensor field satisfies condition (vii).

Remark 3: With contact forces $\bar{f}^+, \bar{f}^- \in L^1([-c, c])$ as in (2.27) the theorem of work and energy reads

$$\begin{aligned} \int_B \underline{E} \cdot \Phi[\underline{E}] da + \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) dq &= \int_{\partial B} \bar{t} \cdot \bar{u} ds + \int_B \bar{b} \cdot \bar{u} da + \\ &\int_{-c}^c (\bar{f}^+ \cdot \bar{u}^+ + \bar{f}^- \cdot \bar{u}^-) dq \end{aligned} \quad (3.5)$$

Consider a set D_c of displacement fields $\bar{u} \in C^2(B \setminus C)$ satisfying (i), (ii) and

$$(iii)_C \quad \bar{e}_2 \cdot (\bar{u}^+ - \bar{u}^-) \geq 0, \quad q \in]-c, c[$$

and assume that

$$\bar{f}^+(q) = -\bar{f}^-(q) = \bar{e}_2 p(q)$$

where the pressure function $p:]-c, c[\rightarrow \mathbb{R}$ satisfies

$$p \in L^1(]-c, c[)$$

$$p(q) \geq 0, \quad q \in]-c, c[\quad (3.6)$$

$$p(q) = 0 \quad \text{if} \quad \bar{e}_2 \cdot (\bar{u}^+(q) - \bar{u}^-(q)) > 0 \quad (3.7)$$

This may represent a situation where, under the loading of the body and the accompanying displacement from the reference configuration, the "crack-surfaces", while separating at some points, may remain in contact at others, transmitting a pressure p . The equations (2.27) may then be written

$$\begin{aligned} \frac{dS^+}{dq} \bar{e}_1 + (I)^+ \bar{e}_2 + p \bar{e}_2 &= \bar{0} \\ \frac{dS^-}{dq} \bar{e}_1 - (T)^- \bar{e}_2 - p \bar{e}_2 &= \bar{0} \end{aligned} \quad (3.8)$$

and furthermore we have

$$\int_{-c}^c (\bar{f}^+ \cdot \bar{u}^+ + \bar{f}^- \cdot \bar{u}^-) dq = \int_{-c}^c p \bar{e}_2 \cdot (\bar{u}^+ - \bar{u}^-) dq = 0 \quad (3.9)$$

and the theorem of work and energy (within the class D_C considered here) is given by (3.1).

4. UNIQUENESS

Any two solutions to the traction problem, satisfying (vii), are equal modulo an (infinitesimal) rigid displacement of B .

Proof: Since (3.1) is established we may proceed as in Kirchhoff's theorem (cf. Knops & Payne [2]).

Let $\bar{u}', \bar{u}''$ ($\bar{E}', \bar{E}'', \bar{I}', \bar{I}'', S^{\pm'}, S^{\pm''}$) be two solutions to the traction problem satisfying (vii). Then $\bar{u} \equiv \bar{u}' - \bar{u}''$ ($\bar{E} \equiv \bar{E}' - \bar{E}'', \bar{I} \equiv \bar{I}' - \bar{I}'', S^{\pm} \equiv S^{\pm'} - S^{\pm''}$) will satisfy the equilibrium equations (2.10) - (2.21) with null body force ($\bar{b} = \bar{0}$) and null boundary tractions ($\bar{t} = 0$). It is not guaranteed that $\bar{u}$ belongs to D (since (iii) may fail). This is however irrelevant in what follows (since (iii) was not used in the proof of (3.1)). Since ($i = 1, 2$)

$$\int_{O_R} \bar{I} \bar{e}_R^i ds = \int_{O_R} (\bar{I}' - \bar{I}'') \bar{e}_R^i ds \leq \int_{O_R} \bar{I}' \bar{e}_R^i ds + \int_{O_R} \bar{I}'' \bar{e}_R^i ds$$

we may conclude, from the theorem of work and energy, that

$$\int_B \bar{E} \cdot \Phi[\bar{E}] da + \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) dq = 0$$

and then from (iv) and (v) we infer that

$$\bar{E}(x) = \bar{0}, \quad x \in B \setminus C \quad (4.1)$$

and thus $\bar{u}$ is an infinitesimal rigid displacement field (cf. Gurtin [5], section 13). Note that from (4.1) we have

$$\frac{du_1^+}{dq} = (\bar{e}_1 \cdot \nabla \bar{u} \bar{e}_1)^+ = (\bar{e}_1 \cdot \bar{E} \bar{e}_1)^+ = 0$$

□

Remark: We may also prove uniqueness of the traction problem within the class D_C .

Let $\bar{u}', \bar{u}''$ ($\bar{E}', \bar{E}'', \bar{I}', \bar{I}'', S^{\pm'}, S^{\pm''}, p', p''$) be two solutions satisfying (vii). Then $\bar{u} \equiv \bar{u}' - \bar{u}''$ ($\bar{E} \equiv \bar{E}' - \bar{E}'', \bar{I} \equiv \bar{I}' - \bar{I}'', S^{\pm} \equiv S^{\pm'} - S^{\pm''}, p = p' - p''$) will satisfy the equations (2.10) - (2.21), with equations (3.8) instead of (2.13), (2.14) and with $\bar{b} = \bar{0}$, $\bar{t} = \bar{0}$. The theorem of work and energy, (3.5), gives

$$\int_B \underline{E} \cdot \nabla[\underline{E}] \, da = \int_{-c}^c (k^+ (\frac{du_1^+}{dq})^2 + k^- (\frac{du_1^-}{dq})^2) \, dq = \int_{-c}^c p \, \bar{e}_2 \cdot (\bar{u}^+ - \bar{u}^-) \, dq \quad (4.2)$$

where

$$\begin{aligned} p \, \bar{e}_2 \cdot (\bar{u}^+ - \bar{u}^-) &= (p' - p'') \, \bar{e}_2 \cdot ((\bar{u}^{'+} - \bar{u}^{''+}) - (\bar{u}^{'+-} - \bar{u}^{''-})) = \\ &= (p' \, \bar{e}_2 \cdot (\bar{u}^{''+} - \bar{u}^{''-}) + p'' \, \bar{e}_2 \cdot (\bar{u}^{'+} - \bar{u}^{'+-})) \end{aligned} \quad (4.3)$$

where we have used (3.7). From (iii)_c, (3.6) and (4.3) we conclude that

$$\int_{-c}^c p \, \bar{e}_2 \cdot (\bar{u}^+ - \bar{u}^-) \, dq \leq 0 \quad (4.4)$$

and uniqueness follows from (4.2), (4.4) and the standard argument.

5. CONCLUDING REMARKS

As we have previously noted, the restrictions in (vii) are somewhat stronger than the ordinary conditions (2.19), (2.20) stated in the traction problem. It would be nice to be able to prove the theorem of work and energy without having to rely on (vii). It seems however that such a proof will call for non-trivial alterations of the simple method used here.

If surface tensions are put equal to zero ($S^\pm = 0$) we obtain the simplified problem:

Find $\bar{u} \in D$ so that

$$\operatorname{div} \bar{I} + \bar{b} = \bar{0}, \quad x \in B \setminus C \quad (5.1)$$

$$\bar{I} = \Phi[\bar{E}] \quad (5.2)$$

$$\bar{E} = \frac{1}{2} (\nabla \bar{u} + \nabla \bar{u}^T) \quad (5.3)$$

$$(\bar{I})^+ \bar{e}_2 = (\bar{I})^- \bar{e}_2 = \bar{0}, \quad q \in]-c, c[\quad (5.4)$$

$$\lim_{R \rightarrow 0} \int_{O_R^i} \bar{I} \bar{e}_R^i ds = \bar{0}, \quad (i = 1, 2) \quad (5.5)$$

$$\lim_{R \rightarrow 0} R \int_{O_R^i} \bar{e}_R^i \times \bar{I} \bar{e}_R^i ds = \bar{0}, \quad (i = 1, 2) \quad (5.6)$$

$$\bar{I} \bar{n} = \bar{t}, \quad x \in \partial B \quad (5.7)$$

Knowles & Pucik [6] have proved the uniqueness of this problem under the assumption that Φ is uniformly positive definite (i.e. there exists a positive constant μ such that $\underline{A} \cdot \Phi(x)[\underline{A}] \geq \mu \underline{A} \cdot \underline{A}$, $x \in B$ and every $\underline{A} \in \operatorname{Sym}(\mathbb{R}^2)$) and that the displacement field is bounded. Conditions (5.5) and (5.6) are absent in their statement of the problem. Note that from (5.1), (5.7) and (2.24) it follows that (5.5) and (5.6) are satisfied automatically.

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